Renormalons Revisited

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Cihan Pazarbaşı

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Outline

Part 1: Renormalons reviewed

- Perturbative series
- Renormalons

Part 2: Renormalons in QM

- Perturbative S-matrix
- Renormalization in QM
- Renormalons in QM

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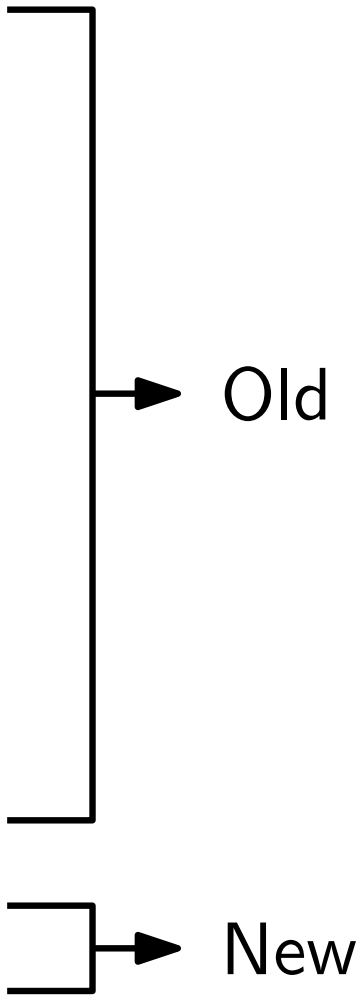

Perturbation theory

$$F(s;\lambda) = F_0(s) + \sum_{n=1}^{\infty} F_n(s)\lambda^n$$

- Technical tool — all over physics, QM
- Definition of the theory — QFT, String th

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Naively

λ small enough $\Rightarrow$ the series 'makes sense'

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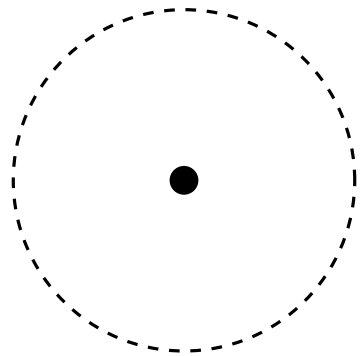

$F(\lambda)$ analytic for all $\lambda \in \mathbb{C}\,,\ |\lambda| < A$

Perturbation theory

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Quantity makes sense at positive and negative coupling

Perturbation theory

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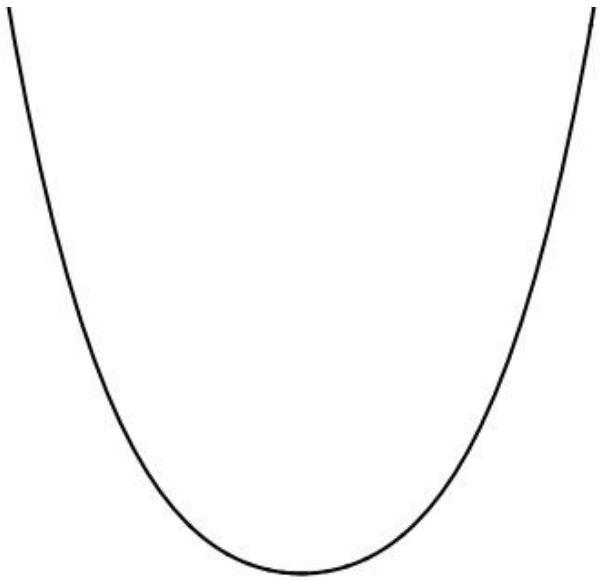
Observation by Dyson:
If quantity makes no sense at negative coupling the perturbation series can **not** be convergent

Perturbation theory

Observation by Dyson:
If quantity makes no sense at negative coupling the perturbation series can **not** be convergent

Example: $V(\lambda) = x^2 + \lambda x^4$



$\lambda > 0$

$\lambda < 0$

Perturbation theory

$$F(s;\lambda) = F_0(s) + \sum_{n=1}^{\infty} F_n(s)\lambda^n$$

Naively

λ small enough $\Rightarrow$ the series 'makes sense'

2) if $F_n \sim A^{-n} n!$ $\Rightarrow$ **asymptotic** series

practical sense: optimal truncation

$$F(\lambda) = \sum_{n=0}^{n_\star} F_n \lambda^n + \mathcal{O}(e^{-\frac{A}{\lambda}}) \qquad n_\star \approx \frac{A}{\lambda}$$

Perturbation theory

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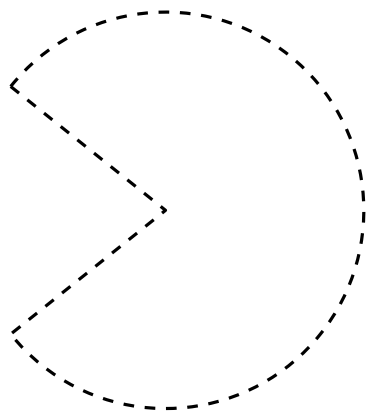
actual sense: Borel resummation

$$F(\lambda) = \int_0^{\infty} dt e^{-t} \sum_{n=0}^{\infty} \frac{F_n}{n!} (\lambda t)^n$$



$F(\lambda)$ is analytic on a wedge

Perturbation theory

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Naively

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$$F(\lambda) = \int_0^{\infty} dt e^{-t} \sum_{n=0}^{\infty} \frac{F_n}{n!} (\lambda t)^n$$

does this really make sense?

Perturbation theory

Borel resummation

$$F(\lambda) = \int_0^\infty dt e^{-t} \sum_{n=0}^\infty \frac{F_n}{n!} (\lambda t)^n$$

Example

$$F_n = A^{-n} n! \qquad \Rightarrow \qquad \sum_{n=0}^\infty \frac{F_n}{n!} (\lambda t)^n = \frac{A}{A - \lambda t}$$

- $A < 0$ Integral well defined
- $A > 0$ Integral not defined

Two options:

$$F_\pm(\lambda) = \lim_{\epsilon \to 0} \int_0^\infty \frac{A e^{-t} dt}{A \pm i\epsilon - \lambda t}$$

Note: $F_+ - F_- = 2\pi i \, A e^{-\frac{A}{\lambda}}$

Non-Perturbative contributions

Asymptotic series with growth $F_n \sim A^{-n} n!$
completed by order $e^{-\frac{A}{\lambda}}$ contributions

non-perturbative

	Origin $n!$ growth	non-perturbative effect
Generically	$\mathcal{O}(n!)$ diagrams of $\mathcal{O}(1)$	instanton(s)

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Renormalon	$\mathcal{O}(1)$ diagrams of $\mathcal{O}(n!)$	unclear

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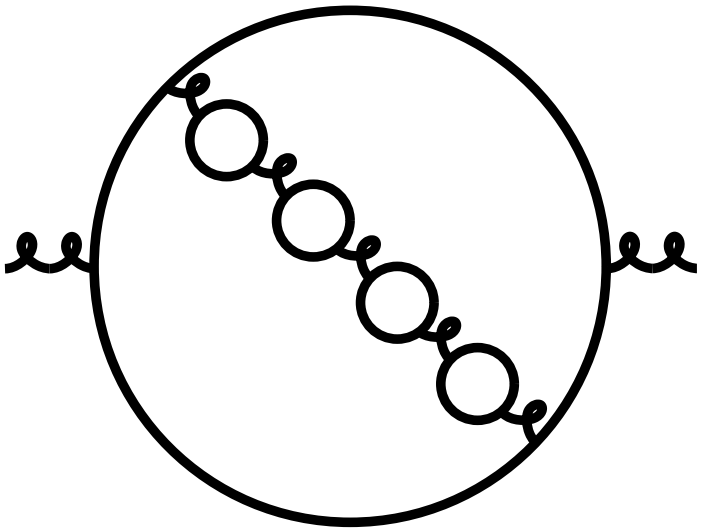
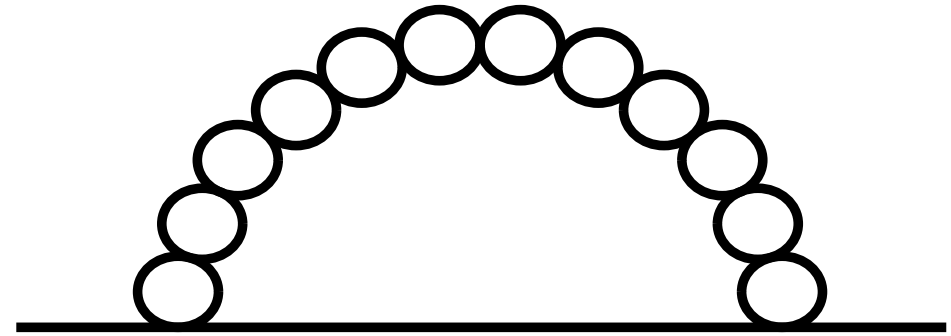
problematic when Borel sum is ambigious
(OPE techniques might solve this)

Renormalons

QCD



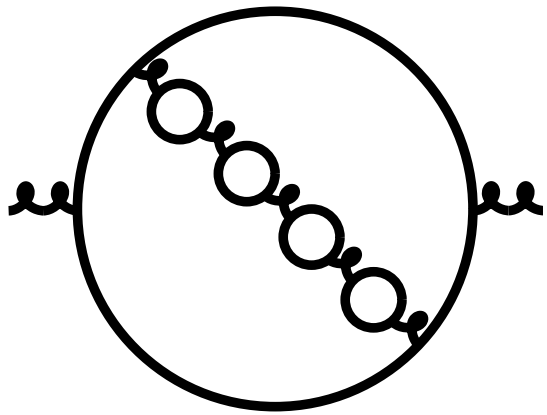
ϕ^4



- renormalized bubble $\sim \log p^2$
- n renormalized bubbles $\sim (\log p^2)^n$
- n renormalized bubbles in a bubble $\sim \int dp^2 f(p^2)(\log p^2)^n$

Renormalons

QCD



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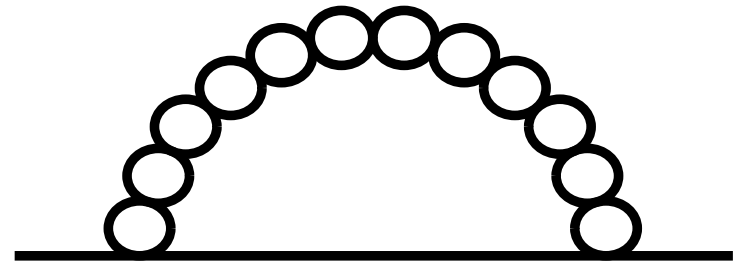

$$\int dp^2 f(p^2)(\log p^2)^n$$

for large n dominated at $p^2 \ll 1$ and $p^2 \gg 1$

IR	UV
$f(p^2) \sim p^{2k+2}$	$f(p^2) \sim p^{-2l-2}$
$k^n \int^\infty dt e^{-t} t^n$	$l^n \int^\infty dt e^{-t} t^n$

$\sim A^n n!$

Renormalons in QM: motivation

Divergences related to 'instantons', examples in

- ODE's
- QM
- QFT

Simple & Rigourous

Complicated & Heuristic

Divergences related to 'renormalons', examples in

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Simple & Rigourous

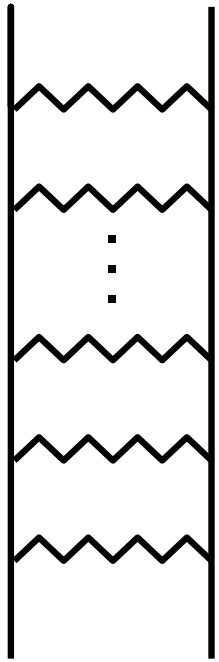
Known non-perturbative definition

Complicated & Heuristic

Perturbative S-matrix

$$H = H_0 + V(X) \qquad H_0 = \frac{P^2}{2m}$$

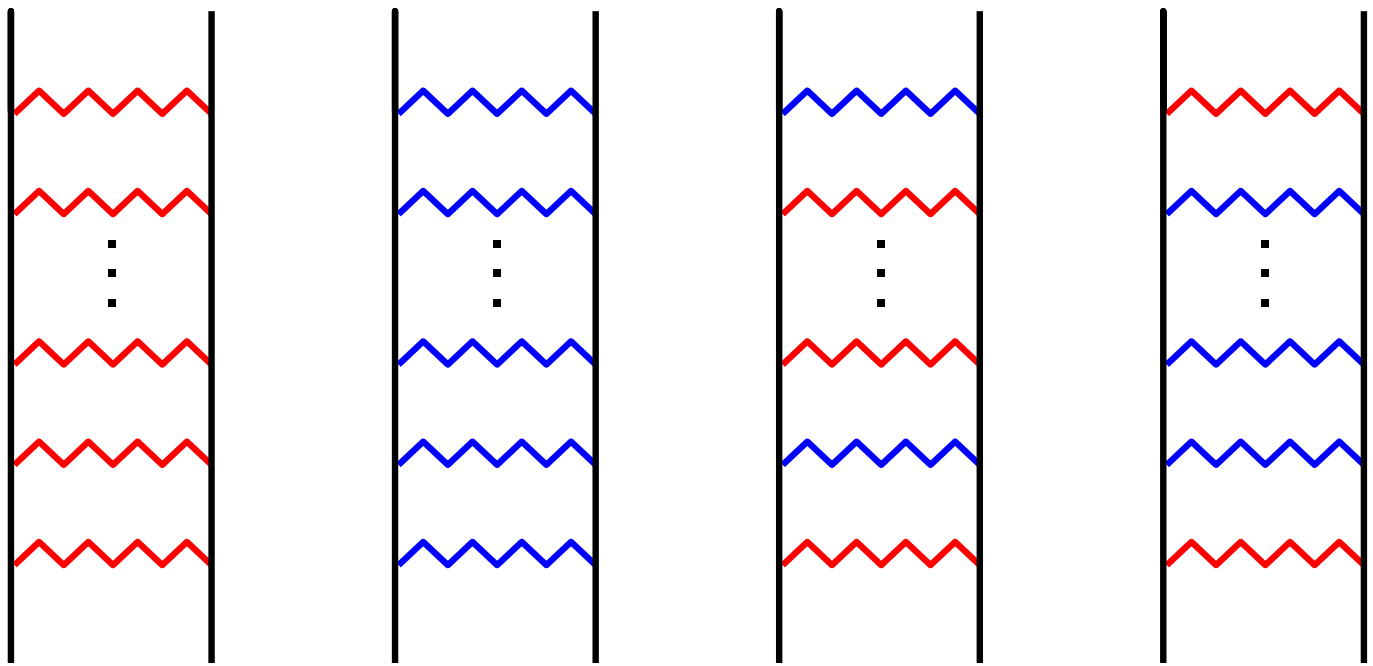
$$S(P_\mathrm{f}, P_\mathrm{i}) = \lim_{t\to\infty} \langle P_\mathrm{f} | e^{iH_0 t_\mathrm{f}} e^{-iH(t_\mathrm{f}-t_\mathrm{i})} e^{-iH_0 t_\mathrm{i}} | P_\mathrm{i} \rangle$$



$$\Leftrightarrow \quad S^{(n)} \sim \int \tilde{V}(P_\mathrm{f} - P_n) \prod_{k=2}^{n} \frac{d^3 P_k}{(2\pi)^3} \frac{\tilde{V}(P_k - P_{k-1})}{E_\mathrm{f} - E_k + i\epsilon}$$

Today's model

$$H = \frac{P^2}{2} + \kappa\delta(x - z) - \lambda_0\delta(x)\delta(y)$$



etc.

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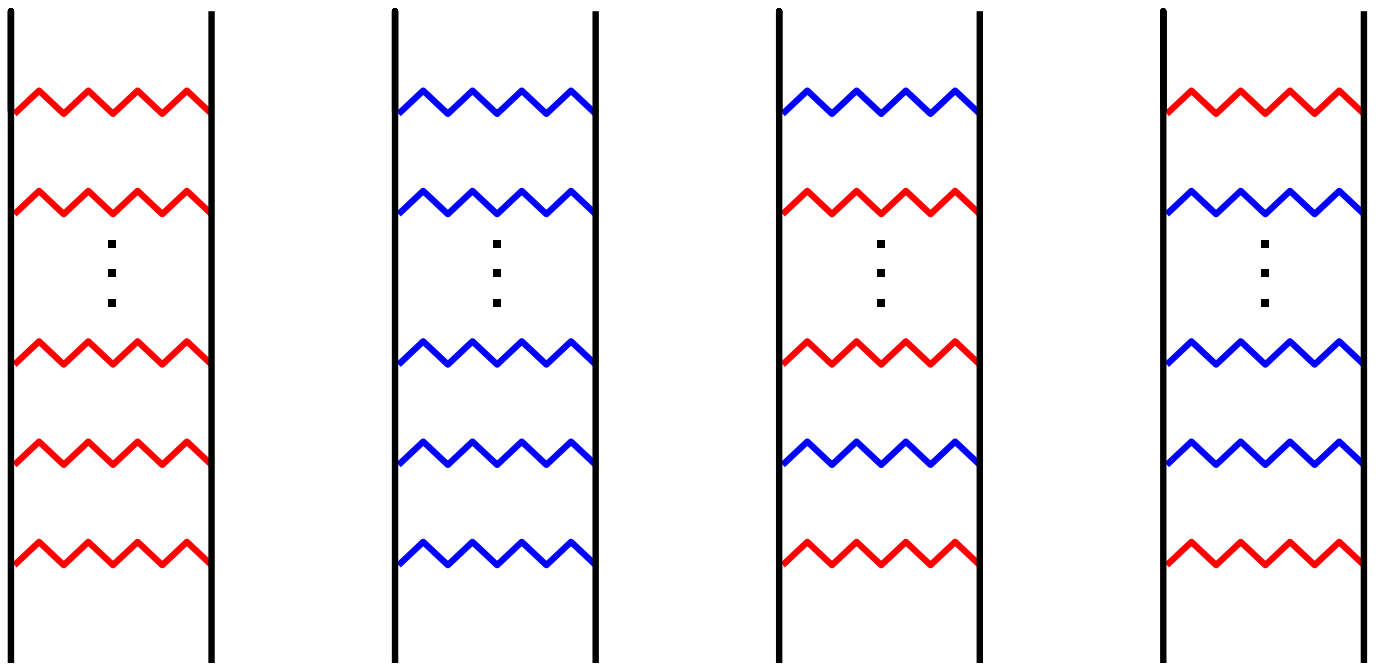
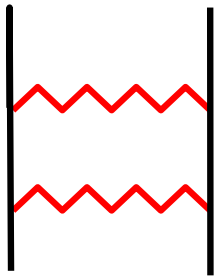


etc.

Bubbles in a bubble !

Warm-up: 1d δ

$$H = \frac{P^2}{2} + \kappa\delta(x - z)$$



$$S^{(2)} \sim \kappa^2 \int \frac{dp_-}{p_{\mathrm{f}-}^2 - p_-^2 + i\epsilon} \sim \frac{\kappa^2}{i|p_{\mathrm{f}-}|}$$

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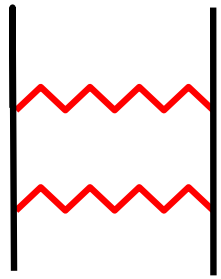

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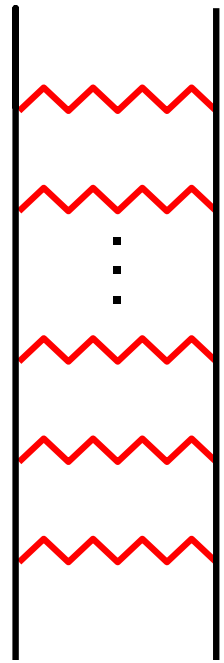

$$\sim \left(\right)^{n-1}$$

$$S^{(n)} \sim \kappa \left(\frac{\kappa}{i|p_{\mathrm{f}-}|} \right)^{n-1}$$

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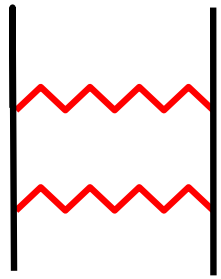

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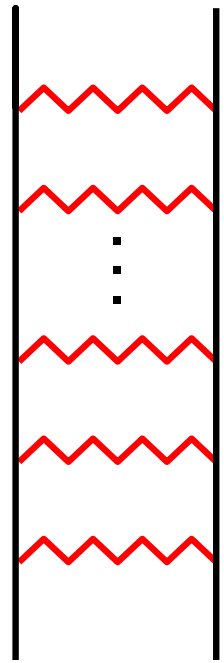

$$\sim \left(\left| \right| \right)^{n-1}$$

$$S^{(n)} \sim \kappa \left(\frac{\kappa}{i|p_{\mathrm{f}-}|} \right)^{n-1}$$

$$\Downarrow$$

$$S \sim \delta(p_{\mathrm{i}-} - p_{\mathrm{f}-}) + \delta(E_{\mathrm{f}} - E_{\mathrm{i}}) \frac{\kappa\,|p_{\mathrm{i}-}|}{i|p_{\mathrm{i}-}| - \kappa}$$

Warm-up: 1d δ

$$H = \frac{P^2}{2} + \kappa\delta(x - z)$$

Exact and fully non-perturbative treatment

- Schrödinger equation on $\mathbb{R}$

$$\psi \sim e^{i|p_-|x_-} + \frac{\kappa}{i|p_-| - \kappa} e^{i|p_- x_-|}$$

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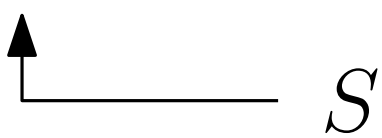

Warm-up: 1d δ

$$H = \frac{P^2}{2} + \kappa\delta(x - z)$$

Exact and fully non-perturbative treatment

- Schrödinger equation on $\mathbb{R}$
- $\mathcal{H} = \mathcal{L}^2(\mathbb{R}_0)$ Free Hamiltonian + s.a. extension

Warm-up: 1d δ

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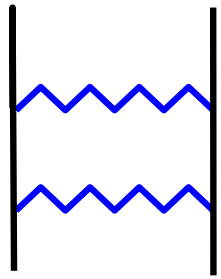
Exact and fully non-perturbative treatment

- Schrödinger equation on $\mathbb{R}$
- $\mathcal{H} = \mathcal{L}^2(\mathbb{R}_0)$ Free Hamiltonian + s.a. extension

Note: $\kappa < 0$, perturbative bound state of $E = -\frac{\kappa^2}{2}$

Renormalization in QM: 2d δ

$$H = \frac{P^2}{2} - \lambda_0 \delta(x)\delta(y)$$



$$S^{(2)} \sim \lambda_0^2 \int_0^\infty \frac{pdp}{p_{\rm f}^2 - p^2 + i\epsilon} \sim \infty$$

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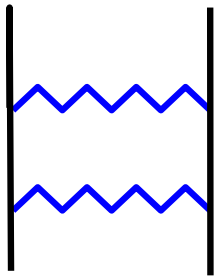

$$S^{(2)} \sim \lambda_0^2 \int_0^{\Omega} \frac{d^2p}{p_{\rm f}^2 - p^2 + i\epsilon} \sim \log\left(\frac{\Omega}{p}\right)$$

- Regularize

Renormalization in QM: 2d δ

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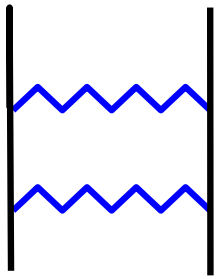

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- Regularize
- Define physical coupling: $S(\mu_\star) = 1 + \lambda_\star$
- Introduce counterterm: $\delta_\lambda \sim \lambda_\star^2 \log \frac{\mu_\star}{\Omega}$

Renormalization in QM: 2d δ

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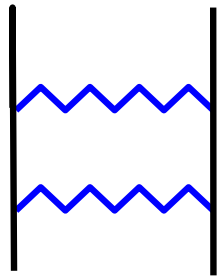

$$S^{(2)} \sim \lambda_\star^2 \log \frac{\mu_\star}{p}$$

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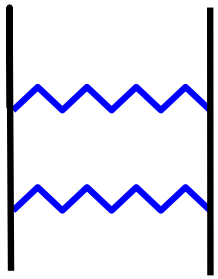

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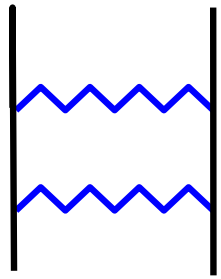

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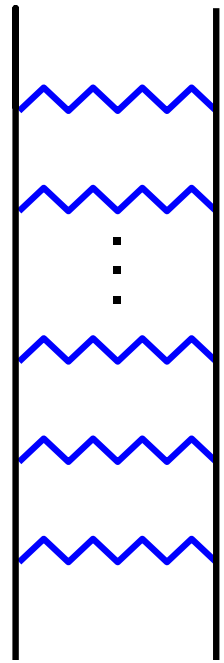


$$\sim \left(\right)^{n-1}$$

$$S^{(n)} \sim \lambda_\star \left(\lambda_\star \log \frac{\mu_\star}{p} \right)^{n-1}$$

$$\Downarrow$$

$$S \sim \delta^2(p_\mathrm{i} - p_\mathrm{f}) + \delta(E_\mathrm{f} - E_\mathrm{i}) \frac{\lambda_\star}{1 + \frac{\lambda_\star}{\pi} \log \frac{p}{\mu_\star}}$$

Renormalization in QM: 2d δ

$$H = \frac{P^2}{2} - \lambda_0 \delta(x)\delta(y)$$

Exact treatment via s.a. extension gives same result

Summary:

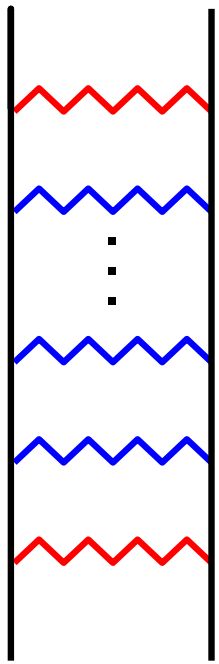
- running coupling

$$\lambda(\mu) = \frac{\lambda_\star}{1+\frac{\lambda_\star}{\pi}\log\frac{\mu}{\mu_\star}} = \frac{\pi}{\log\frac{\mu}{\Lambda}}$$

- asymptotically free
- 'dynamically generated' scale $\Lambda = \mu_\star e^{-\frac{\pi}{\lambda_\star}}$
- non-perturbative bound state $E = \frac{\Lambda^2}{2m}$

Renormalons in QM

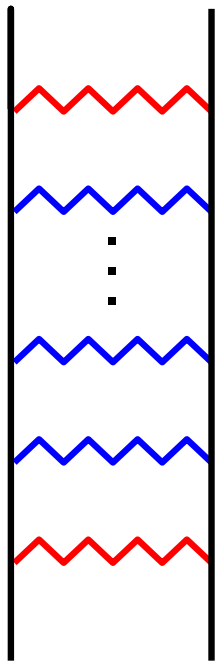
$$H = \frac{P^2}{2} + \kappa\delta(x-z) - \lambda_0\delta(x)\delta(y)$$



$$S \ni \kappa^2\lambda_\star^{n+1}\int_{-\infty}^{\infty} \frac{du\ \left(\log\frac{2E_{\rm f}-u^2}{\mu_\star}\right)^n}{((p_{\rm f}-u)(u-u_{\rm f})+i\epsilon)((p_{\rm i}-u)(u-u_{\rm i})+i\epsilon)}$$

Renormalons in QM

$$H = \frac{P^2}{2} + \kappa\delta(x-z) - \lambda_0\delta(x)\delta(y)$$



$$S \ni \kappa^2\lambda_\star^{n+1}\int_{-\infty}^{\infty}\frac{du\ \left(\log\frac{2E_{\rm f}-u^2}{\mu_\star}\right)^n}{((p_{\rm f}-u)(u-u_{\rm f})+i\epsilon)((p_{\rm i}-u)(u-u_{\rm i})+i\epsilon)}$$

UV, $u\to\infty$ $\quad\approx\quad (\frac{2}{3})^n n!$

IR, $u\to\sqrt{2mE_{\rm f}}$ $\quad\approx\quad (-1)^n n!$

Conclusion

Renormalons in QM appear to exist

- This simple model can probably be exactly solved
- Non-perturbative interpretation of these renormalons?
- Lessons for QFT?