Gravitational Collapse of Thin Shell of Dust in Shape Dynamics[a]

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[a] based on arXiv:1804.02902

Fundamental Principles

Mach's Principle

a) The application of Mach's principle is to be considered whenever direct observations or theoretical considerations suggest that the physical configuration space of a closed dynamical system is to be obtained by group quotienting of a larger configuration space that contains redundant information unobtainable by direct observation;

b) once the quotiented configuration space $\mathcal{Q}$ has been selected, the dynamical theory defined on it is to be such that either 1) specification of an initial point $q \in \mathcal{Q}$ together with a *direction* d in $\mathcal{Q}$ at q defines a unique curve in $\mathcal{Q}$ as the evolution of the system given the directly observable initial data q, d or 2) instead of a point q and a direction, a point and a *tangent vector* at q are specified.

- Ref. [1]

Mach's Principle

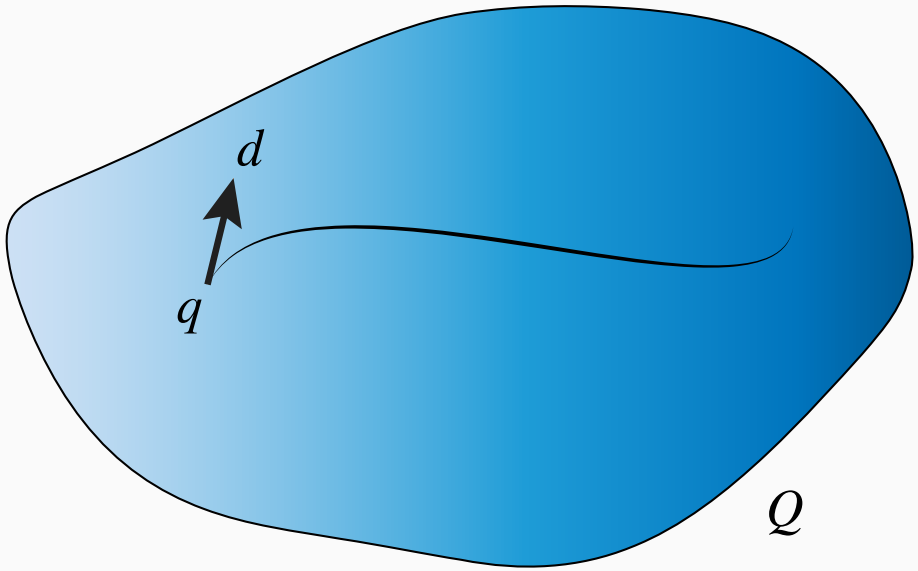


Figure 1: A point in reduced configuration space and a direction determines the fate of evolution in shape dynamics. This is the strong version of the Mach's Principle.

Mach's Principle ⟶ Mach-Poincaré Principle

Mach-Poincaré Principle

Barbour and Bertotti has given a precise formulation of the Mach's Principle in the form of Mach-Poincaré Principle [3]:

> *Physical (or relational) initial configurations and their first derivatives alone should determine uniquely the future evolution of the system.*

Shape Dynamics

What is Shape Dynamics?

- Shape dynamics (SD) is a theory of gravitation.
- A point q and a direction d in the reduced configuration space determines the evolution of universe: It is Machian.
- Local translations, rotations, dilatations are all *gauge*.
- It locally agrees with GR, departs from GR in global aspects [5].
- SD and GR share the same reduced configuration space.

A Toy Model for Shape Dynamics

Let us consider an N-body problem in the case of Newtonian gravitational attraction between the bodies [2] as a toy model for SD. (The SD part of this presentation is heavily influenced by the excellent SD tutorial [5].)

There is no time!

We begin with a quote from Ernst Mach [4]:

> *It is utterly beyond our power to measure the changes of things by time. Quite the contrary, time is an abstraction at which we arrive through the changes of things.*

In SD, we think of time as an *emergent phenomenon*.

Counting the # of dof's of kinematic states

- Each particle has 3 coordinates in space: $3N$ dof.
- We work in the center of mass frame, the total momentum is zero: -3 dof.
- There is no absolute space. Hence the universe has a vanishing total angular momentum: -3 dof.
- Only the dimensionless ratios of inter-particle distances play role. The size of the system is unimportant: -1 dof.

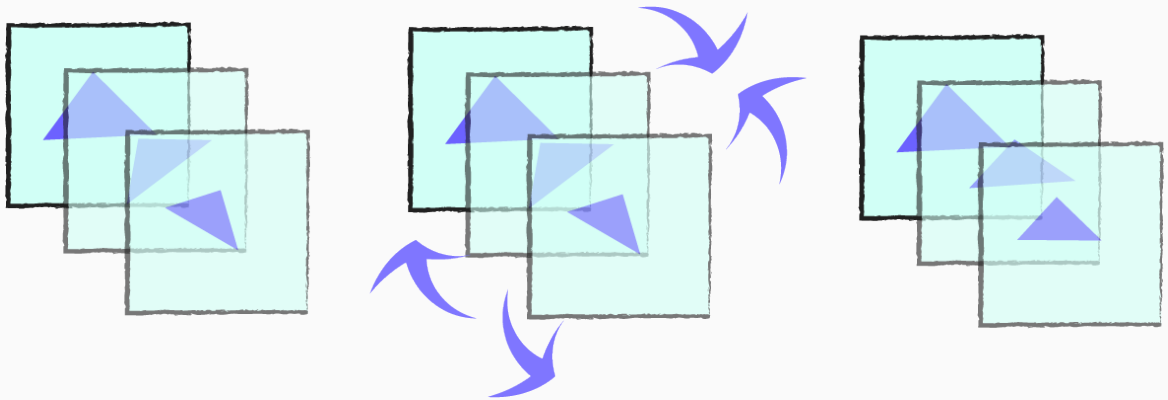
In total each *snapshot* of the universe contain $3N - 7$ fundamental dof.

Best Matching

Best matching describes a metric in shape space (reduced conf. spc.) (modification made on [5] version):

$$d_{bm}(q^i, q^f) = \inf_{\Omega,\theta,\phi} \left(\sum_{a=1}^{N} |r_a^i - \phi \Omega_{ab} r_b^f - \theta_a| \right)^{1/2}, \tag{1}$$

where $\phi \in \mathbb{R}^+, \Omega \in SO(3), \theta \in \mathbb{R}^3$.



Best Matching

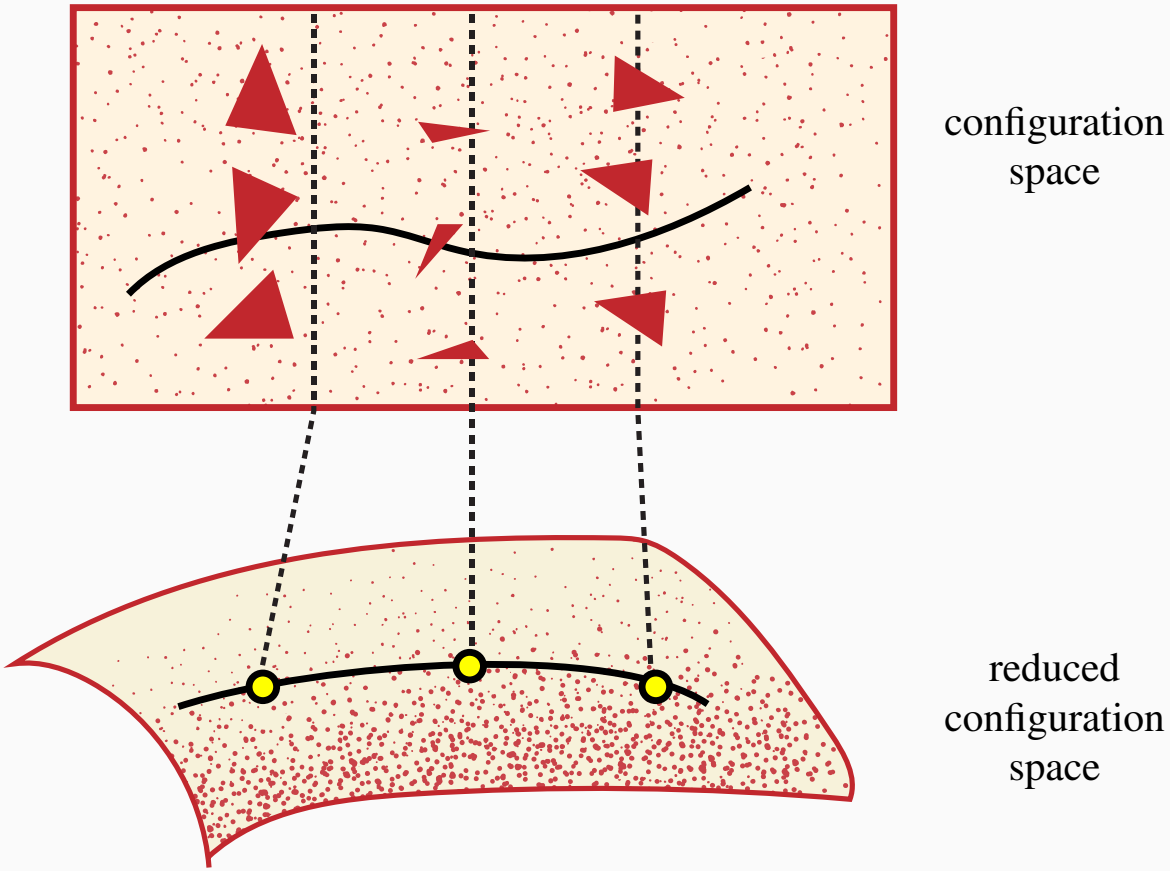


Figure 2: Gauge orbits are vertical lines in the configuration space. Horizontal displacement in the configuration space is true change.

Full SD

In full SD,

- primitive ontology: 3-conformal geometries
- there is local Weyl symmetry
- space-time is emergent

Vacuum Variables

The space we consider is $[0,1] \times S^2$: A filled-in unit ball. The most general spherically symmetric metric and metric momentum can be written as follows:

$$g_{ab} = \mathrm{diag}(\mu^2, \sigma, \sigma \sin^2\theta), \tag{2}$$

$$p^{ab} = \mathrm{diag}\left(\frac{f}{\mu}, \frac{s}{2}, \frac{s}{2\sin^2\theta}\right) \sin\theta, \tag{3}$$

where μ, σ, f, s are function of the radial variable r.

SD Constraints in CMC Gauge

In the CMC (Constant Mean extrinsic Curvature) gauge, the vacuum SD constraints are as follows [5]:

$$\mathcal{H} = \frac{1}{\sqrt{g}}\left(p^{ab}p_{ab} - \frac{1}{2}p^2\right) - \sqrt{g}R, \tag{4}$$

$$\mathcal{H}_a = -2\nabla_b p^b_a, \tag{5}$$

$$p = \sqrt{g}\langle p\rangle, \tag{6}$$

where $\mathcal{H}$ is the ADM Hamiltonian, $\mathcal{H}_a$ is the diffeomorphism constraint and p is the trace of the conjugate metric momentum.

When the values of g_{ab} and p^{ab} is put in equations (4), (5) and (6) we get the following equations [5]:

$$-\frac{1}{6\sigma\mu^2}\Big[\sigma^2\mu s^2+4f^2\mu^3-4f\sigma\mu^2 s+12\sigma\mu\sigma'' \\ -12\sigma\sigma'\mu'-3\mu(\sigma')^2-12\sigma\mu^3-\langle p\rangle^2\sigma^2\mu^3\Big]=0, \tag{7}$$

$$\mu f'-\frac{1}{2}s\sigma'=0, \tag{8}$$

$$\mu f+s\sigma-\langle p\rangle\mu\sigma=0, \tag{9}$$

where $(')$ stands for differentiation with respect to r.

Using equations (5) and (6) gives us:

$$\frac{\mu}{\sqrt{\sigma}}\left(f\sqrt{\sigma}-\frac{1}{3}\langle p\rangle\sigma^{3/2}\right)'=0. \tag{10}$$

Hence we obtain the solution as:

$$f\sqrt{\sigma}-\frac{1}{3}\langle p\rangle\sigma^{3/2}=A(\tau), \tag{11}$$

where τ is York time.

The Hamiltonian constraint can be written as follows [5]:

$$-\frac{\sigma^{1/2}\mu}{\sigma'}\frac{\partial}{\partial r}\Big[\left(\frac{\sigma'}{\sigma^{1/4}\mu}\right)^2-4\sqrt{\sigma}-\frac{f^2}{\sqrt{\sigma}}\Big] \tag{12}$$

$$=\frac{2f\mu}{\sigma'}\left(f'+\frac{f\sigma'}{2\sigma}-\frac{\langle p\rangle\sigma'}{2}\right). \tag{13}$$

The term inside the parenthesis on the right hand side of equation (13) vanishes [5], hence the term whose radial derivative is taken must equal to a constant [5]:

$$\left(\frac{\sigma'}{\sigma^{1/4}\mu}\right)^2-4\sqrt{\sigma}-\frac{f^2}{\sqrt{\sigma}}=-8m. \tag{14}$$

When the solution (11) is put into equation (14) we obtain the μ^2 as:

$$\mu^2 = (\sigma')^2\Big(A^2\sigma^{-1} + (\frac{2}{3}\langle p\rangle A - 8m)\sigma^{1/2} + 4\sigma + \frac{1}{9}\langle p\rangle^2\sigma^2\Big)^{-1} \quad (15)$$

We have found μ^2 as in (15) however we know that it should be greater than zero because the three dimensional metric should be regular. This condition puts a restriction on the radial derivative of σ (σ') and the σ itself. This issue is handled in [5] in detail and we suggest readers read this note.

Coupling a Spherically Symmetric Thin Shell of Dust to SD

We suppose there are n particles each of mass m_0. The ADM constraints for a single particle is as follows [6]:

$$\mathcal{H} = \delta(x^a - y^a)\sqrt{g^{ab}p_a p_b + m_0^2}, \tag{16}$$

$$\mathcal{H}_a = \delta(x^a - y^a)p_a. \tag{17}$$

If we take the continuum limit of particles on a shell of radius r_s, the constraints become [6]:

$$\mathcal{H} = \sqrt{h}\ \rho(r_s)\ \delta(r - r_s)\ \sqrt{g^{rr}p_r^2 + m_0^2}, \tag{18}$$

$$\mathcal{H}_a = \delta_a^r\ \sqrt{h}\ \rho(r_s)\ \delta(r - r_s)\ p_r, \tag{19}$$

where $\rho(r_s)$ is a scalar function yet to be determined and h_{ij} is the metric on S^2 induced by g_{ij}.

Imposing the condition that in this process the number of particles is constant, gives us [6]:

$$\rho(r_s) \int d\theta d\phi \sqrt{h(r_s)} = 4\pi n. \tag{20}$$

If we rescale the momentum and mass as $p_s = np_r$ and $M = nm_0$, the constraints (7), (8) and (9) become [5]:

$$\begin{aligned} &- \frac{1}{6\sigma\mu^2}\Big[\sigma^2\mu s^2 + 4f^2\mu^3 - 4f\sigma\mu^2 s + 12\sigma\mu\sigma'' \\ &\quad - 12\sigma\sigma'\mu' - 3\mu(\sigma')^2 - 12\sigma\mu^3 - \langle p\rangle^2\sigma^2\mu^3\Big] \\ &= \delta(r - r_s)\sqrt{p_s^2/\mu^2 + M^2}, \end{aligned} \tag{21}$$

$$\mu f' - \frac{1}{2}s\sigma' = -\frac{p_s}{2}\delta(r - r_s), \tag{22}$$

$$\mu f + s\sigma - \langle p\rangle\mu\sigma = 0. \tag{23}$$

CMC constraint (23) gives us $s = \langle p \rangle \mu - \mu f/\sigma$ and when put in (22) yields (There is a typo in [5]. The factor before $\langle p \rangle \sigma$ should be $-1/2$ not $-1/3$.):

$$\frac{\mu}{\sqrt{\sigma}} \left(f\sqrt{\sigma} - \frac{1}{3}\langle p \rangle \sigma^{3/2} \right)' = -\frac{p_s}{2}\delta(r - r_s). \tag{24}$$

A method to solve (24) is given in [6, 5], the solution is:

$$f(r) = \frac{1}{3}\langle p \rangle \sigma + \frac{A_-}{\sqrt{\sigma}}\Theta(r_s - r) + \frac{A_+}{\sqrt{\sigma}}\Theta(r - r_s), \tag{25}$$

where Θ is the Heaviside distribution and

$$A_+ - A_- = -\frac{p_s\sqrt{\sigma(r_s)}}{2\mu(r_s)}. \tag{26}$$

Here A_- and A_+ are functions of r_s and p_s.

On the other hand, if we look at (21), the only singular part on the left hand side comes from the $-2\sigma''/\mu$ and we have [5]:

$$\operatorname{sing}\left(\frac{2\sigma''}{\mu}\right) = -\delta(r - r_s)\sqrt{p_s^2/\mu^2 + M^2}, \tag{27}$$

which yields

$$\operatorname{sing}(\sigma'') = -\frac{1}{2}\delta(r - r_s)\sqrt{p_s^2 + M^2\mu^2}. \tag{28}$$

Let $\gamma = \lim_{r\to r_s^+} \sigma'$ and $\kappa = \lim_{r\to r_s^-} \sigma'$, then we have [5]:

$$\gamma - \kappa = -\frac{1}{2}\sqrt{p_s^2 + M^2\mu^2(r_s)}. \tag{29}$$

Time Evolution of the Shell

The presymplectic form is as follows [6]:

$$\theta = \int dr d\theta d\phi p^{ab} \delta g_{ab} + 4\pi p_s \delta r_s. \tag{30}$$

When definitions of metric and conjugate momentum of metric which are defined in (2) and (3) are taken into account in equation 30 and the angular part is integrated out, what we have is the following:

$$\theta = 4\pi \int_0^1 dr(2f\delta\mu + s\delta\sigma) + 4\pi p_s \delta r_s. \tag{31}$$

Using the CMC condition (23) we find:

$$\theta = 4\pi \int_0^1 dr \left(\langle p \rangle \mu \delta\sigma - \frac{2\mu}{\sqrt{\sigma}} \delta(f\sqrt{\sigma}) \right) + 4\pi p_s \delta r_s. \tag{32}$$

Let us use the form of f found in (25), then we obtain:

$$\theta = -\underbrace{\frac{8\pi}{3} \int_0^1 dr \mu\sigma\delta\langle p \rangle}_{\equiv\phi} - 8\pi \left[\delta A_- \int_0^{r_s} dr \frac{\mu}{\sqrt{\sigma}} + \delta A_+ \int_{r_s}^1 dr \frac{\mu}{\sqrt{\sigma}} \right]. \tag{33}$$

Here we move on to the isotropic coordinates: $\mu = \sqrt{\sigma}/r$.

In this case, the presymplectic potential becomes $\theta = -\phi + 8\pi\ \delta(A_+ - A_-)\ln r_s$ modulo an exact form. Using equation (26) we can write this as follows:

$$\theta = -\phi - 4\pi\ \delta(p_s r_s)\ln r_s. \tag{34}$$

The symplectic form, ω, which is minus the exterior derivative of θ is found out to be:

$$\omega = -\delta\theta = \delta\phi + 4\pi\ \delta r_s \wedge \delta p_s, \tag{35}$$

where $\delta\phi$ does not depend on δr_s or δp_s.

In matrix form, ω can be represented as follows:

$$\omega_{ab} = \begin{pmatrix} F(\delta\phi) & 0 & 0 \\ 0 & 0 & 4\pi \\ 0 & -4\pi & 0 \end{pmatrix}, \tag{36}$$

where the order of the coordinates is as "others", r_s, p_s where F is some function of $\delta\phi$. The inverse of the symplectic form to calculate the Poisson brackets is found as follows:

$$\omega^{ab} = \begin{pmatrix} F^{-1}(\delta\phi) & 0 & 0 \\ 0 & 0 & -1/4\pi \\ 0 & 1/4\pi & 0 \end{pmatrix}. \tag{37}$$

$\mathcal{H}$ is the rescaled Hamiltonian density in equation (21). The Hamiltonian is then found out to be the following:

$$\begin{aligned} H &= \int dr d\theta d\phi \sqrt{g} \mathcal{H} \\ &= \pi^2 \sigma(r_s) \sqrt{p_s^2 + M^2 \mu^2(r_s)}. \end{aligned} \tag{38}$$

Now, we find the equations of motion for r_s and p_s. We have already calculated the symplectic form so we can calculate the Poisson brackets of r_s and p_s with the Hamiltonian. For $\dot{r}_s$ we have:

$$\begin{aligned}
\dot{r}_s &= \{r_s, H\}, \\
&= \frac{1}{4\pi}\frac{\partial H}{\partial p_s}, \\
&= \frac{\pi}{4}\sigma(r_s)\frac{p_s}{\sqrt{p_s^2 + M^2\mu^2(r_s)}}, \\
&= \frac{\pi}{4}\sigma(r_s)\frac{p_s}{\sqrt{p_s^2 + M^2\sigma(r_s)/r_s^2}}.
\end{aligned} \tag{39}$$

And for $\dot{p}_s$ we have the following:

$$
\begin{aligned}
\dot{p}_s &= \{p_s, H\}, \\
&= -\frac{1}{4\pi}\frac{\partial H}{\partial r_s}, \\
&= -\frac{\pi}{4}\sigma'(r_s)\sqrt{p_s^2 + M^2\mu^2(r_s)} \\
&\quad - \frac{\pi}{4}\sigma(r_s)\frac{M^2\mu(r_s)\mu'(r_s)}{\sqrt{p_s^2 + M^2\mu^2(r_s)}}, \\
&= -\frac{\pi}{4}\sigma'(r_s)\sqrt{p_s^2 + M^2\sigma(r_s)/r_s^2} \\
&\quad - \frac{\pi}{4}\frac{\sigma(r_s)}{2r_s^2}\frac{M^2}{\sqrt{p_s^2 + M^2\sigma(r_s)/r_s^2}} \\
&\quad \times \left(\sigma'(r_s) - \frac{2\sigma(r_s)}{r_s}\right).
\end{aligned}
\tag{40}
$$

These equations are very complex to be solved analytically. We will make an approximation and suppose that the shell is very close to origin: $r_s \ll 1$.

Perturbation Analysis

Near the origin the metric should be flat and for that purpose $\sigma(r) \approx r^2$. There is one more issue to be solved: $\sigma'(r)$ is discontinuous at $r = r_s$. Therefore we will equate $\sigma'(r_s)$ to the average of left (κ) and right (γ) limits of $\sigma'(r_s)$. Since we can approximate $\sigma(r) \approx r^2$ inside r_s, we find $\kappa = 2r_s$. The use of equation (29) yields:

$$\sigma'(r_s) = \frac{\kappa + \gamma}{2} \approx -\frac{1}{4}\sqrt{p_s^2 + M^2} + 2r_s. \tag{41}$$

For $r_s \ll 1$, the equations of motion (39) and (40) become:

$$\dot{r}_s \approx \frac{\pi}{4} \frac{r_s^2 p_s}{\sqrt{p_s^2 + M^2}}, \tag{42}$$

$$\dot{p}_s \approx \frac{\pi}{16}(p_s^2 + M^2) - \frac{\pi}{2} r_s \sqrt{p_s^2 + M^2} + \frac{\pi M^2}{32}. \tag{43}$$

These equations admit a static solution: $r_s = 3M/16,\ p_s = 0$.

If we do a perturbation analysis around the static solution the results turn out to be functions of circular functions:

$$r_s(t) \approx \frac{3M}{16} + A_r \sin\left(\Omega(t - t_r)\right), \tag{44}$$

$$p_s(t) \approx A_p \sin\left(\Omega(t - t_p)\right), \tag{45}$$

where $\Omega = 3\pi M^{3/2}/32\sqrt{2}$, A_r, A_p, t_r, t_p are integration constants and there is a condition on the first two integration constants: $A_r \ll 3M/16 \ll 1, A_p \ll 1$.

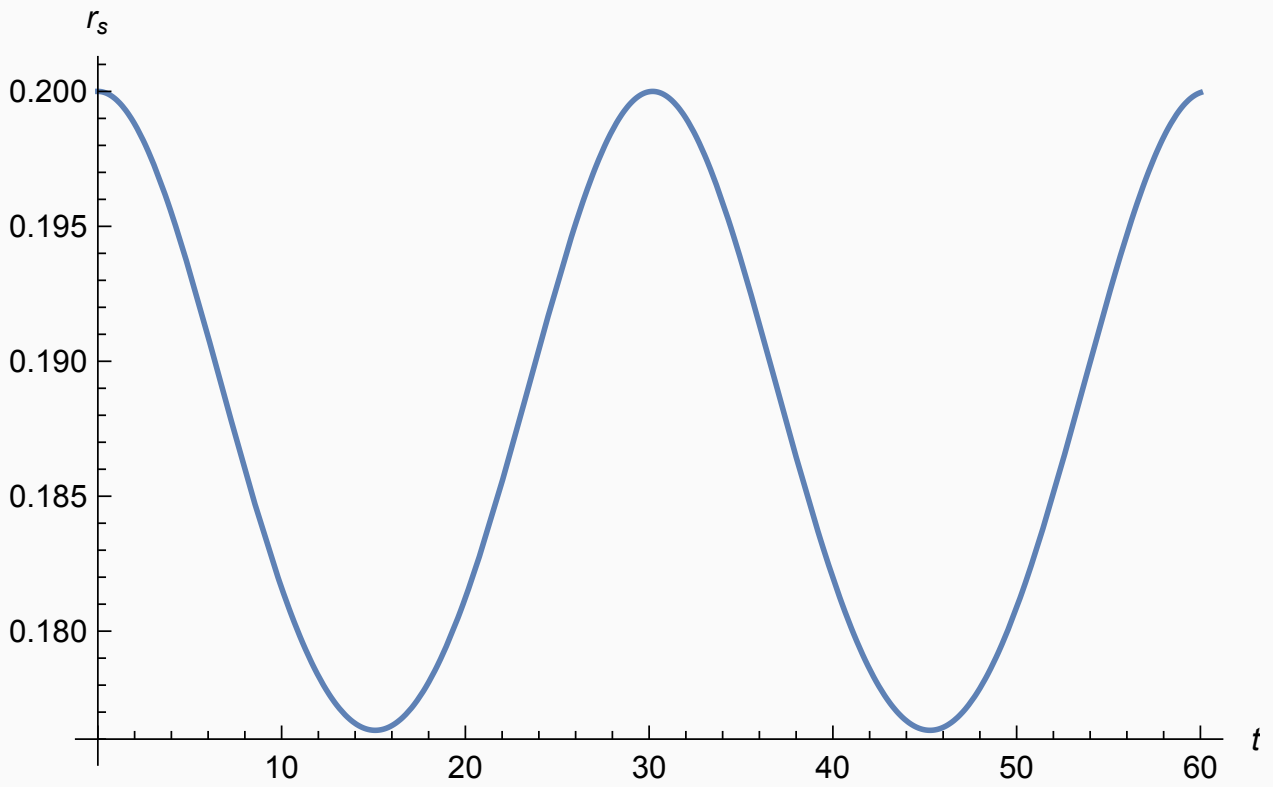


Figure 3: The evolution of r_S for $r_S(0) = 0.2$, $p_S(0) = 0$ and $M = 1$.

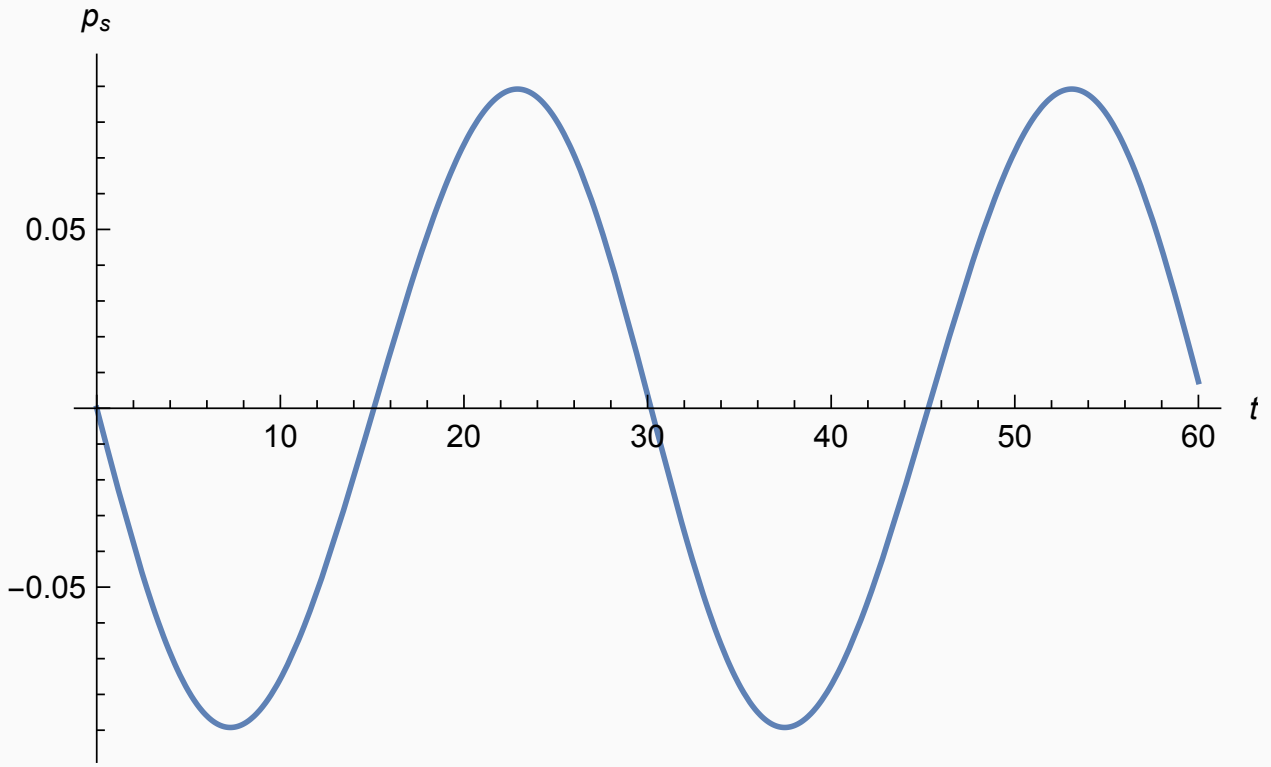


Figure 4: The evolution of p_s for $r_s(0) = 0.2$, $p_s(0) = 0$ and $M = 1$.

High Momentum Limit: $-p_s \gg M$

In the limit $-p_s \gg M$, the equations of motion become the following two:

$$\dot{r}_s \approx -\frac{\pi r_s^2}{4}, \tag{46}$$

$$\dot{p}_s \approx \frac{\pi p_s^2}{16}. \tag{47}$$

The solutions are given as follows:

$$r_s(t) \approx \frac{1}{1/r_s(0) + \pi t/4}, \tag{48}$$

$$p_s(t) \approx \frac{1}{1/p_s(0) - \pi t/16}. \tag{49}$$

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