Wilsonian Renormalization as a Quantum Circuit

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Recent Trends in String Theory and Related Topics
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  - Quantum Circuit Complexity: Holographic Complexity, Complexity in QFTs, ...

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- Introducing *entanglement dof's*, TNs offer concrete ways to bring in the affect of far dof's into correlations inside a local region

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- The simplest case is 2d CFT. If we pick up a random state in the Hilbert space of a lattice version, the entropy of a block is proportional to the block size but the entanglement entropy scales with the logarithm of the block size which is exponentially smaller.
- Physical states of interest leave in a corner of the Hilbert space with exponentially suppressed entropy corresponding to entanglement between subsystems.

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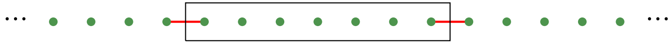

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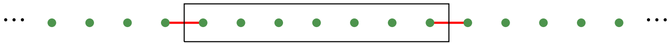

- But even a generalized version of such a state does not respect translational symmetry

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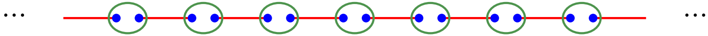
- We introduce 'partons' for each dof



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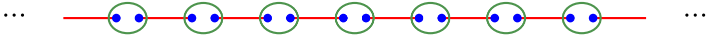

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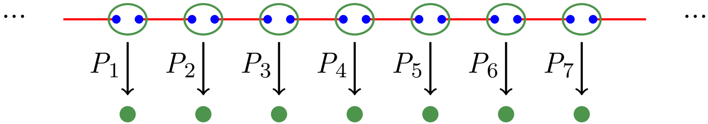

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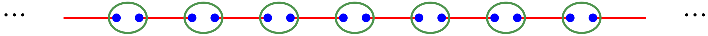

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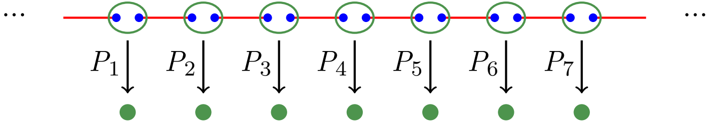

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- We have constructed the state as

$$|\Psi\rangle = \bigotimes_{i=1}^{N} P_i \cdot |b\rangle^{\otimes N}$$

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$$P_i = \sum_{\substack{\alpha_i,\beta_i=1,\cdots,D \\ r=1,\cdots,d}} A^{(i),r}_{\alpha_i\beta_i}|r\rangle\langle\alpha_i,\beta_i| \quad , \quad |b\rangle = \frac{1}{\sqrt{D}}\sum_{k=1}^{D}|k,k\rangle$$

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- The number of parameters instead of d^N is linear in N, namely

$$N \cdot d \cdot D^2$$

Outline

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- The two key tools in this talk are:
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- The two key tools in this talk are:
  - Quantum Circuits (a sequence of unitaries)
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- Ideas:
  1. **Reformulation of Dynamics:** One may think about unitary dynamics as a quantum circuit which entangles local dof's of a state
  2. **Reformulation of Renormalization Group:** One can think of renormalization group flow as a sequence of local gates which disentangle and isometries which coarse grain local dof's of a state

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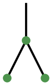

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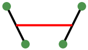

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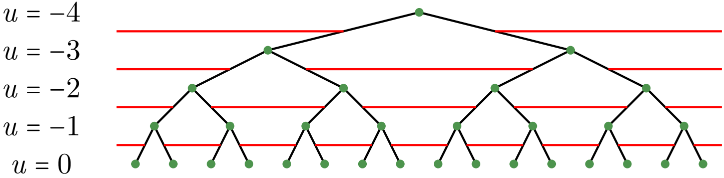

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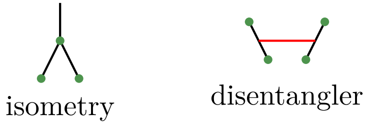

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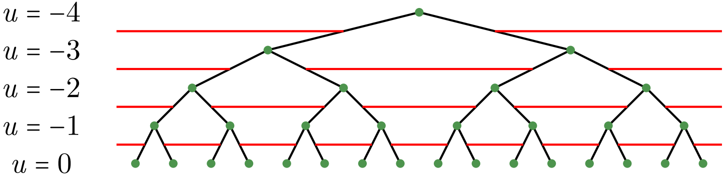

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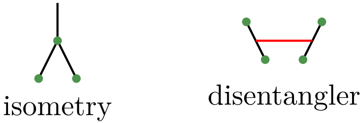

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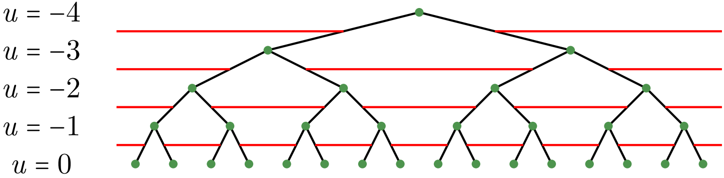

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- The entanglement free state where the layered structure ends is called $u = u_{\mathrm{IR}}$

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- **Question:** How to find $K(u)$?

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  2. Develop an analytic approach (quantum circuit perturbation theory)
  3. Express Wilsonian RG as spatially local cMERA circuit (it turns out that perturbative Wilsonian RG yields a local $K(s)$)

Spatial Wilsonian RG

- The Hilbert space

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- The partition function

$$Z^{\Lambda}[\{J_i(\vec{p})\}] = \lim_{T \to \infty(1-i\epsilon)} \int \prod_{|\vec{p}| \leq \Lambda} \mathcal{D}\phi \, \mathcal{D}\pi \, e^{i \int_{-T}^{T} dt \left(\int^{\Lambda} d^d\vec{p} \left[\pi\dot{\phi} - \mathscr{H}^{\Lambda}\right] \right)} e^{-i \int^{\Lambda} d^d\vec{p} \, \sum_i J_i(\vec{p}) \mathcal{O}_i(\vec{p}, 0)}$$

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- (Weyl-ordered) Bender-Dunne basis ['89]:

$$T_{m,n} := \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} x^k p^m x^{n-k} = \frac{1}{2^m} \sum_{j=0}^{m} \binom{m}{j} p^j x^n p^{m-j}$$

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$$U = \exp\left\{ i \sum_{m,n=0}^{\infty} c_{m,n}\, T_{m,n} \right\}$$

- **Q:** When is the RHS a sum of finitely many terms?

$$\exp\left\{ i \sum_{m,n=0}^{N_1} c^{(1)}_{m,n}\, T_{m,n} \right\} \exp\left\{ i \sum_{m,n=0}^{N_2} c^{(2)}_{m,n}\, T_{m,n} \right\} = \exp\left\{ i \sum_{m,n=0}^{\infty} c_{m,n}\, T_{m,n} \right\}$$

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  we can determine Q if all Q_i's are at most quadratic.

- **Q:** How to deal with higher orders?
- **A:** A very special case (order: largest $m + n$)

$$[\text{order } 2, \text{order } k] \leq \text{order } k$$

- This means

$$e^{i(\,Q^{(1)}_2 + \epsilon Q^{(1)}_k\,)}\, e^{i(\,Q^{(2)}_2 + \epsilon Q^{(2)}_k\,)} \,=\, e^{i(\,Q_2 + \epsilon Q_k\,)} + \mathcal{O}(\epsilon^2)$$

Continuum Circuits & Perturbation Theory

- Consider

$$U = \exp\left(-i \sum_{0 \le p+q \le M} c_{p,q}\, T_{p,q}\right)$$

but with the following form

$$U = \exp\left(-i\left(\sum_{p+q \le 2} c_{p,q}\, T_{p,q} + \epsilon \sum_{2 < r+s \le M} c_{r,s}\, T_{r,s}\right)\right)$$
$$\equiv \exp\left(-i\left(Q_2 + \epsilon Q_{\text{higher}}\right)\right)$$

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- It is not hard to show that

$$U = \left(1 + \epsilon\, \frac{e^{-i\,\mathrm{ad}_{Q_2}} - 1}{\mathrm{ad}_{Q_2}}\, Q_{\text{higher}}\right) e^{-i\,Q_2} + \mathcal{O}(\epsilon^2)$$

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- Remember that the $\mathcal{O}(\epsilon)$ term is at most of order M

cMERA: free massive scalar theory

- Starting with

$$\langle \phi | \Omega \rangle = \mathcal{N} \exp\left(-\frac{1}{2} \int d^d\vec{x}\, \phi(\vec{x})\, M\, \phi(\vec{x}) \right)$$

- Determine what $K(s)$ needs to be such that

$$\langle \phi | \mathcal{P}_s\, e^{-i \int_{-\infty}^{u} ds\, (K(s)+L)} | \Omega \rangle =$$

$$\mathcal{N} \exp\left\{ -\frac{1}{2} \int^{\Lambda} d^d\vec{k}\, \phi(\vec{k})\, \sqrt{\vec{k}^2 + e^{-2u} m^2}\, \phi(-\vec{k}) - \underbrace{\frac{1}{2} \int_{\Lambda}^{\infty} d^d\vec{k}\, \phi(\vec{k})\, M\, \phi(-\vec{k})}_{\text{leftover, unused modes}} \right\}$$

cMERA: free massive scalar theory

- We find

$$M = \sqrt{\Lambda^2 + m^2}$$

$$K(s) = \int d^d\vec{k} \left[\frac{1}{4}\theta\left(1 - \frac{|\vec{k}|}{\Lambda}\right) - \frac{1}{8}\log\left(\frac{\vec{k}^2 + e^{-2s}m^2}{\Lambda^2 + m^2}\right)\frac{|\vec{k}|}{\Lambda}\,\theta'\left(1 - \frac{|\vec{k}|}{\Lambda}\right)\right] \times \left[\phi(\vec{k})\pi(-\vec{k}) + \pi(\vec{k})\phi(-\vec{k})\right]$$

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- first term: squeeze on all modes for $|\vec{k}| \leq \Lambda$
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- second term: peaked around $|\vec{k}| = \Lambda$,
in position space: non-negligible for scales $\Lambda^{-1} \sim (me^{-u})^{-1}$
- Different from the previous one [Haegeman et. al. '11]: it gives the right state at *any* scale

(d+1)-dim φ^4 Theory

- Lets focus on

$$H = \int d^d x \left[\frac{1}{2} \left(\hat{\pi}(x)^2 + \hat{\phi}(x) \left(-\nabla^2 + m^2 \right) \hat{\phi}(x) \right) + \frac{\lambda}{4!} \hat{\phi}(x)^4 \right]$$

- In momentum space we have

$$H^\Lambda = \frac{1}{2} \int^\Lambda d^d k \left(\hat{\pi}(k)\hat{\pi}(-k) + \hat{\phi}(k) \left(k^2 + m^2 \right) \hat{\phi}(-k) \right) \\ + \frac{\lambda}{4!} \frac{1}{(2\pi)^d} \int^\Lambda d^d k_1 \, d^d k_2 \, d^d k_3 \, \hat{\phi}(k_1)\hat{\phi}(k_2)\hat{\phi}(k_3)\hat{\phi}(-k_1 - k_2 - k_3)$$

Vacuum State Wave-functional of φ^4 Theory

- The $\mathcal{O}(\lambda)$ vacuum wave-functional is given by [Hatfield '91]

$$\langle \phi | \Psi_\Lambda \rangle = \mathcal{N} \exp\left(-G[\phi] - \delta m^2 R_1[\phi] - \lambda R_2[\phi] \right) + \mathcal{O}(\lambda^2)$$

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where

$$G[\phi] = \frac{1}{2}\int^\Lambda d^d k\,\phi(k)\,\omega_k\,\phi(-k), \qquad \delta m^2 = \frac{\lambda}{2}\int_{\Lambda e^u}^{\Lambda}\frac{d^d\vec{p}}{(2\pi)^d}\frac{1}{\vec{p}^2+m^2},$$

$$R_1[\phi] = \frac{1}{4}\int^\Lambda d^d k\,\frac{1}{\omega_k}\,\phi(k)\phi(-k),$$

$$R_2[\phi] = \frac{1}{16}\int^\Lambda d^d k\,\frac{1}{\omega_k}\left(\int^\Lambda\frac{d^d q}{(2\pi)^d}\frac{1}{\omega_k+\omega_q}\right)\phi(k)\phi(-k)$$

$$+\frac{1}{24}\frac{1}{(2\pi)^d}\int^\Lambda\frac{d^d k_1\,d^d k_2\,d^d k_3}{\omega_{k_1}+\omega_{k_2}+\omega_{k_3}+\omega_{-k_1-k_2-k_3}}\,\phi(k_1)\phi(k_2)\phi(k_3)\phi(-k_1-k_2-k_3)$$

What is the Corresponding Circuit?

- Non-trivial kernels are

$$
\begin{aligned}
K(s) = &\int d^d\vec{x}_1\, d^d\vec{x}_2\, f_{2,0}(\vec{x}_1,\vec{x}_2;s)\left[\phi(\vec{x}_1)\pi(\vec{x}_2)+\pi(\vec{x}_1)\phi(\vec{x}_2)\right] \\
&+\lambda\int d^d\vec{x}_1\, d^d\vec{x}_2\, f_{2,1}(\vec{x}_1,\vec{x}_2;s)\left[\phi(\vec{x}_1)\pi(\vec{x}_2)+\pi(\vec{x}_1)\phi(\vec{x}_2)\right] \\
&+\lambda\int d^d\vec{x}_1\, d^d\vec{x}_2\, d^d\vec{x}_3\, d^d\vec{x}_4 \times
\end{aligned}
$$

$$
\Big(f_4^{(1)}(\vec{x}_1,\vec{x}_2,\vec{x}_3,\vec{x}_4;s)\left[\phi(\vec{x}_1)\pi(\vec{x}_2)\pi(\vec{x}_3)\pi(\vec{x}_4)+\pi(\vec{x}_2)\pi(\vec{x}_3)\pi(\vec{x}_4)\phi(\vec{x}_1)\right]
$$

$$
+ f_4^{(3)}(\vec{x}_1,\vec{x}_2,\vec{x}_3,\vec{x}_4;s)\left[\pi(\vec{x}_1)\phi(\vec{x}_2)\phi(\vec{x}_3)\phi(\vec{x}_4)+\phi(\vec{x}_2)\phi(\vec{x}_3)\phi(\vec{x}_4)\pi(\vec{x}_1)\right]\Big)
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- The explicit forms are long (see our paper)

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- The explicit forms are long (see our paper)
- (Using Fourier-analytic arguments) We prove that they are local

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Important (perturbative) Lesson

Spatial Wilsonian RG = Local cMERA circuit

Summary & Outlook

- Summary
  - TN's as powerful tools to study low energy physics
  - MPS as a pedagogical example to see how the scaling of parameters suppresses from exponential to linear

  - Quantum circuit perturbation theory
  - Work out cMERA kernels exactly up to any order in perturbation theory
  - Explicitly for d-dim φ^4 scalar theory
  - Wilsonian RG can be re-expressed as local cMERA in position space
- Outlook
  - (Non-perturbative) numerical approach for ground state of field theories
  - cMERA circuit for fermions and (non-)Abelian gauge theories
  - In principle application to strongly interacting theories