Asymptotic Symmetries of Maxwell theory at Spatial Infinity in Arbitrary Dimensions

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based on 1902.02769

Ultimate task

To understand/decide what happens far away from a Maxwell system and to find symmetry generators.

Symmetries

The set of all ways of changing your point of view which leave the appearance of something unchanged is called the symmetry group.

Symmetry in physical systems

- Invariance of Largangian $\quad \mathcal{L} \to \mathcal{L} + \partial_\mu \mathcal{K}^\mu$
- Invarince of action $\quad S \to S + \int_\partial \mathcal{K}$
- A vector field preserving the Symplectic form $\quad L_\xi \Omega = 0$
- A phase space function G commuting with the Hamiltonian $\{H, G\} \approx 0$

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Example: Phase rotation of a Dirac particle

$$G = e \int d^3x \psi^\dagger \psi$$

The Charge is the on-shell value of the generator.

Asymptotic symmetries

Gauge variables are determined with respect to reference frames.

Gauge symmetry:

- is the freedom to change the reference frame arbitrarily,
- is generated by *constraints* of the theory,
- has vanishing charge.

Regge-Teitelboim '74

Addition of appropriate surface term makes the generators well-defined and the charges non-vanishing.

Example: $U(1)$ gauge symmetry of charged Dirac particle

$$G = \int d^3x \lambda(x)(\partial_i \pi^i + e\psi^\dagger \psi) - \int_\partial n_i \pi^i$$

where $\lambda|_\partial = 1$.

Recent motivation

- Infinite dimensional group of symmetries
  Strominger et al '14, Campiglia et al '15

$$G_\lambda \approx -\int_\partial n_i \lambda(\theta, \varphi)\pi^i$$

- (Quantized) symmetry generators commute with S matrix

$$\langle out|GS - SG|in\rangle = 0 \quad \Rightarrow \quad \textit{Soft photon theorem.}$$

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Led to

- New soft theorems Sen et al '17,
- Conformal description of S-matrix Pasterski-Shao '17,
- New explanation of IR divergences Strominger'17.
- Understanding memory effects, old and new Gibbons '17
- Hope in resolving information paradox Hawking et al '16

Higher dimensional Maxwell theory

Rationale

Boundary behavior is sensitive to spacetime dimension.

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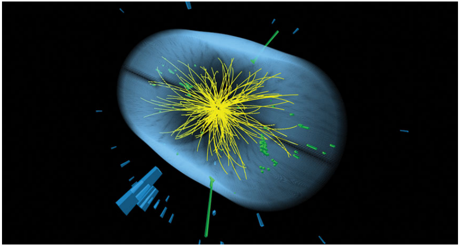
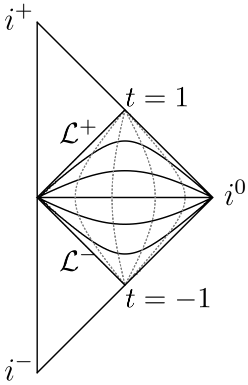
Boundary behavior is sensitive to spacetime dimension.

- $D \geq 4$ at null infinity, memory and symmetries,
  Wald-Satishchandran 1901.04467
- $D \geq 3$ at spatial infinity, covariant hyperbolic approach,
  EE,1902.02769
- $D \geq 5$ at null infinity, soft theorem and symmetries,
  He-Mitra,1903.02608
- $D \geq 5$ at spatial infinity, Hamiltonian approach,
  Henneaux-Troessaert, 1903.04437

Spatial Infinity

The *collider dark age* when everything has gone inside but nothing has come out.





The theory

$$S = \int_{M_d} \sqrt{g} \left(-\frac{1}{4} \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu} + \mathcal{A}_\mu J^\mu \right) \, ,$$

Gauge symmetry

$$\mathcal{A}_\mu \to \mathcal{A}_\mu + \partial_\mu \Lambda$$

The action is invariant up to boundary terms

$$\delta_\lambda S = \int_{\partial M_d} \sqrt{g} \Lambda n_\mu J^\mu$$

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Boundary conditions: To admit arbitrary configurations of moving electric charges. The starting point:

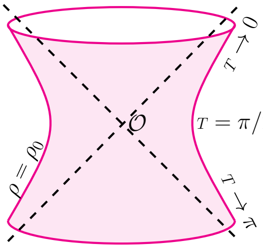
$$\mathcal{F}_{tr} = \frac{1}{r^{2-d}} \qquad \text{electric charge at the origin}$$

Hyperbolic coordinates

$$ds^2 = d\rho^2 + \overbrace{\frac{\rho^2}{\sin^2 T}\left(-dT^2 + q_{AB}dx^A dx^B\right)}^{\text{metric on } dS_{d-1}}, \qquad 0 \le T \le \pi\,.$$

where

$$\begin{cases} \rho^2 = r^2 - t^2 \\ \cos T = t/r \end{cases} \qquad\qquad \begin{cases} t = \rho \cot T \\ r = \rho / \sin T \end{cases}$$



Field equations

$SO(d-1,1)$ representations: $\quad \mathcal{F}_{\mu\nu} \quad \rightarrow \quad \mathcal{F}_{a\rho} \quad \oplus \quad \mathcal{F}_{ab}$

Find the boundary conditions by applying the Poincare generators on the static charge

- Boosts:

$$\mathcal{F}_{T\rho} = \rho^{3-d} \sin^{d-3} T \sim \rho^{3-d} \quad \Rightarrow \quad \mathcal{F}_{a\rho} \sim \rho^{d-3}$$

  and still $\mathcal{F}_{ab} = 0$

- Translations:

$$\mathcal{F}_{\rho a} \sim \mathcal{O}(\rho^{3-d}) \,, \quad \Rightarrow \quad \mathcal{F}_{ab} \sim \mathcal{O}(\rho^{3-d})$$

Boundary conditions

From

$$\mathcal{F}_{\rho a} \sim \mathcal{O}(\rho^{3-d}) \quad , \quad \mathcal{F}_{ab} \sim \mathcal{O}(\rho^{3-d})$$

boundary conditions on the gauge field follow

$$\mathcal{A}_a \sim \mathcal{O}(\rho^{3-d}) \qquad \mathcal{A}_\rho \sim \mathcal{O}(\rho^{3-d})$$

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Boundary conditions

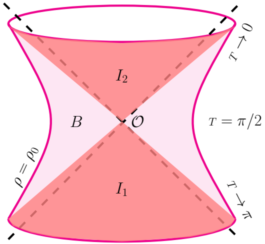
We allow *pure gauge* fluctuations at the boundary

$$A_\mu = \partial_\mu \phi + \mathcal{O}(\rho^{3-d}) \qquad \phi \sim \mathcal{O}(1)$$

Side effect: The action principle is down.

The action principle

$$S = \int_{M_d} \sqrt{g} \left(-\frac{1}{4} \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu} + \mathcal{A}_\mu J^\mu \right) ,$$



On the timelike boundary: $\int_B \sqrt{h}\, \delta A_b^{(0)} F_{(d-3)}^{b\rho}$

The action principle

From boundary conditions $\mathcal{A}_a \sim \mathcal{A}_\rho \sim \mathcal{O}(\rho^{3-d})$ an important fact follows

$$F_{a\rho} = \partial_a \psi \qquad \text{and } \psi \text{ is gauge invariant.}$$

So the boundary term becomes

$$\int_B \sqrt{h}\, \delta A_b^{(0)} F^{b\rho}_{(d-3)} = \int_B \sqrt{h}\, \delta A_b^{(0)} \partial^b \psi = \int_B \sqrt{h} D^b \left(\delta A_b^{(0)} \psi \right) + \cdots$$

The boundary term becomes a total divergence by fixing the *Lorenz gauge* asymptotically

$$D^a A_a^{(0)} + \alpha(d-2) A_\rho^{(1)} = 0\,, \qquad \alpha = 1$$

Action

If $\delta A_\mu^{(0)}$ is not fixed, the action principle is well-defined only if the Lorenz gauge is imposed at leading order.

Conserved Charges

Gauge tansformation of the action+equations of motion+boundary conditions+ gauge fixing:

$$0 = \delta_\lambda S - \int_{I_2} \Lambda J^T + \int_{I_1} \Lambda J^T \approx - \int \sqrt{h}\,\left(-\lambda \mathcal{F}^{\rho T} + \partial^T \lambda \psi\right)\Big|_{\partial I_1}^{\partial I_2}$$

where

$$\delta A_a^{(0)} = \partial_a \lambda\,, \qquad D^a D_a \lambda = 0\,,$$

We identify the conserved charge

$$Q_\lambda = \int_{\partial I} \sqrt{g}\,(\lambda \mathcal{F}^{T\rho} - \partial^T \lambda \psi)$$

The Asymptotic Symmetry Group

Field equation: $D^a D_a \psi(x^b) = 0$
Gauge fixing: $D^a D_a \lambda(x^b) = 0$
The general solution for $D^a D_a f(x^b) = 0$ is

$$f(y, \hat{x}) = (1 - y^2)^{\frac{d-2}{4}} \sum_{\ell=1} Y_\ell(\hat{x}) \left(a_\ell P^{(d-2)/2}_{(2l+d-4)/2}(y) + b_\ell Q^{(d-2)/2}_{(2l+d-4)/2}(y) \right) ,$$

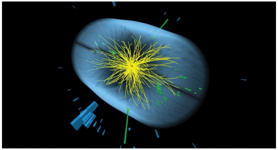
There are two sets of solutions for λ and ψ. The charge

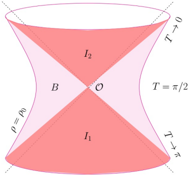
$$Q_\lambda = \int_{S^{d-2}} \sqrt{h} \, (\lambda \partial^T \psi - \partial^T \lambda \psi)$$

is the Wronskian of the differential equation, non-zero only if

$$\lambda \to P \; \psi \to Q \qquad \text{or} \qquad \lambda \to Q \; \psi \to P$$

The antipodal matching





Expanding $D_a D^a f = 0$ near the future boundary of dS_{d-1}, there are two asymptotic fall-off s for the solutions

$$f_- = T^{d-2}\bar{\psi}(\hat{x}) + \mathcal{O}(T^d)\,, \qquad f_+ = \bar{\lambda}(\hat{x}) + \mathcal{O}(T^2)\,.$$

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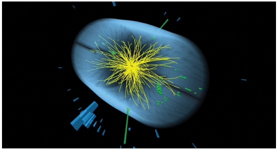

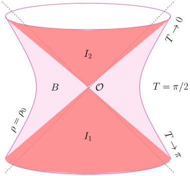

Expanding $D_a D^a f = 0$ near the future boundary of dS_{d-1}, there are two asymptotic fall-off s for the solutions

$$f_- = \tau^{d-2}\bar{\psi}(\hat{x}) + \mathcal{O}(\tau^d)\,, \qquad f_+ = \bar{\lambda}(\hat{x}) + \mathcal{O}(\tau^2)\,.$$

Antipodal boundary condition

$$\psi(\tau, \hat{x}) = -\psi(\pi - \tau, -\hat{x})\,.$$

The antipodal matching: justification

- Required by Soft theorems in 4d
  Strominger et al '14
- Respected by the electromagnetic field of moving charges
- Required to make the symplectic form finite,
  Henneaux-Troessaert '18

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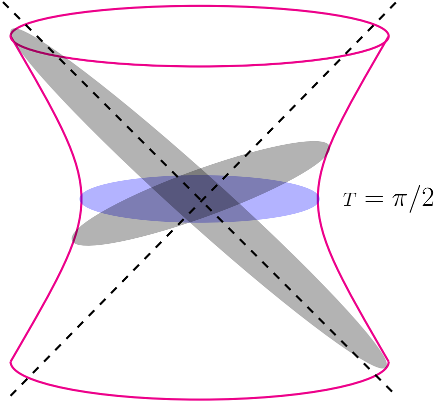
- Regularity of the field strength at light-cone

$$
\begin{aligned}
f_- :&\quad F_{ru} \sim r^{2-d} \\
f_+ :&\quad F_{ru} \sim \begin{cases} (ur)^{(2-d)/2} & d > 4 \\ \frac{\ln r}{ur} & d = 4 \end{cases}
\end{aligned}
$$

Minkowski Observers



Near the *equator*,

$$t = \rho \cot T \cong \rho(\frac{\pi}{2} - T)\,, \qquad\qquad r = \frac{\rho}{\sin T} \cong \rho\,.$$

Minkowski Observers

Components contributing to the charge

$$\bar{A}_r(\hat{x}) = \psi(\frac{\pi}{2}, \hat{x}) \qquad \pi^r \equiv \sqrt{g} F^{rt} = -\sqrt{q}\, \partial_T \psi(\frac{\pi}{2}, \hat{x})\,,$$

where

$$A_r = \partial_r \lambda + r^{3-d} \bar{A}_r(\hat{x}) + \mathcal{O}(r^{2-d})\,, \qquad \lambda \sim \mathcal{O}(r^0)\,.$$

The leading pure gauge components are

$$A_t(\hat{x}) = \frac{1}{r} \partial_T \phi(\frac{\pi}{2}, \hat{x})\,, \qquad A_B(\hat{x}) = \partial_B \phi(\frac{\pi}{2}, \hat{x})$$

Now defining

$$\lambda(\hat{x}) \equiv \lambda(\frac{\pi}{2}, \hat{x})\,, \qquad \mu(\hat{x}) \equiv \partial_T \lambda(\frac{\pi}{2}, \hat{x})\,.$$

The leading gauge transformations are

$$A_B(\hat{x}) + \partial_B \lambda(\hat{x}) \qquad A_t(\hat{x}) \to A_t(\hat{x}) + \frac{1}{r} \mu(\hat{x})$$

Minkowski Observers

Parity conditions

$$\bar{A}_r(\hat{x}) = -\bar{A}_r(-\hat{x}) \qquad \pi^r(\hat{x}) = +\pi^r(-\hat{x}) \qquad A_B(\hat{x}) = -A_B(-\hat{x})$$

$$\mu(\hat{x}) = -\mu(-\hat{x}) \qquad \lambda(\hat{x}) = +\lambda(-\hat{x})$$

The conserved charge

$$Q_\lambda = -\int_{S^{d-2}} \sqrt{q}\,\left(\lambda\pi^r - \mu\bar{A}_r\right)$$

Summary

- Physical boundary condition for Maxwell theory, plus large pure gauge variations
- An asymptotic gauge fixing is needed to make the action principle well-defined
- The charges can be found directly from gauge invariance of the action
- The antipodal condition ensures regularity at light cone

Thank you

Three dimensional Theory

$$\mathcal{F}_{a\rho} \sim \mathcal{O}(\rho^0)\,, \qquad \mathcal{F}_{bc} \sim \mathcal{O}(\rho^0)\,. \tag{1}$$

$$\mathcal{A}_{\rho} \sim \mathcal{O}(\rho^0)\,, \qquad \mathcal{A}_{a} \sim \mathcal{O}(\rho^0)\,. \tag{2}$$

$$D_a D^a \psi = 0\,. \tag{3}$$

$$d\tilde{s}^2 = \frac{-d\tau^2 + d\varphi^2}{\sin^2 \tau} = \frac{\mathrm{d}x^+ \mathrm{d}x^-}{\sin^2 \tau}. \tag{4}$$

$$\partial_+ \partial_- \psi = 0\,. \tag{5}$$

$$\psi(\tau, \varphi) = a_0 + b_0 \tau + \sum_{n \neq 0} \left(a_n e^{inx^+} + b_n e^{inx^-} \right) \tag{6}$$

$$Q_\lambda = \int_{S^1} \sqrt{h}\, (\partial_\tau \lambda \psi - \lambda \partial_\tau \psi) \tag{7}$$

$$\psi(\tau, \varphi) = -\psi(\pi - \tau, \varphi + \pi) \tag{8}$$

The antipodal map $(\tau, \varphi) \to (\pi - \tau, \varphi + \pi)$ is equivalent to $x^+ \leftrightarrow x^-$. As a result, (6) is divided into even and odd parts

$$\psi(\tau, \varphi) = c_0 T + \sum_{n\neq 0} \frac{c_n}{n} e^{in\varphi} \sin nT\,, \qquad c_n = c^*_{-n} \qquad \text{odd} \tag{9a}$$

$$\lambda(\tau, \varphi) = d_0 + \sum_{n\neq 0} d_n e^{in\varphi} \cos nT\,, \qquad d_n = d^*_{-n} \qquad \text{even} \tag{9b}$$