THE PARITY ODD STRUCTURE OF CONFORMAL FIELD THEORIES IN $D = 3$

Based on

1707.03007, JHEP 1711 (2017) 171,
1812.07774, JHEP 1904 (2019)023

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INTRODUCTION & MOTIVATION

- Conformal field theories are highly constrained by symmetry.

Using conformal symmetry to study these theories one can restrict our attention to conformal primaries.

• Conformal primaries, one is interested is their conformal dimension $\Delta_{\mathcal{O}}$

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle = \frac{1}{|x_1 - x_2|^{2\Delta_{\mathcal{O}}}}$$

and their three point functions.

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle =$$

$$\frac{C_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_2}}{|x_1 - x_2|^{\Delta_{\mathcal{O}_1}+\Delta_{\mathcal{O}_2}-\Delta_{\mathcal{O}_3}}|x_2 - x_3|^{\Delta_{\mathcal{O}_2}+\Delta_{\mathcal{O}_3}-\Delta_{\mathcal{O}_1}}|x_3 - x_1|^{\Delta_{\mathcal{O}_3}+\Delta_{\mathcal{O}_1}-\Delta_{\mathcal{O}_2}}}$$

• Can one constrain the allowed conformal dimensions $\Delta_{\mathcal{O}}$ or the OPE coefficients $C_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_2}$ that appear in conformal field theories.

- Consider $d = 2$ CFT.

Using the Virasoro (conformal) algebra and positivity of the norm.
Conformal dimension of any primary

$$\Delta \geq 0.$$

The central charge

$$c \geq 0.$$

The central charge is the coefficient that appears in the 3 point function of stress tensor of the theory.

$$\langle T(z_1)T(z_2)T(z_3)\rangle = \frac{c}{(z_1 - z_2)^2(z_2 - z_3)^2(z_3 - z_4)^2}$$

- All conformal field theories admit a stress tensor $T_{\mu\nu}$.

It is a conformal primary with conformal dimension $\Delta_T = d$, for conformal field theories in d space-time dimensions.

Conformal symmetry constrains the three point function of stress tensors to a few un-determined numbers.

• Similarly if a conformal field theory admits a conserved $U(1)$ current j_μ .

It is a conformal primary of dimensions $d-1$ for conformal field theories in d space-time dimensions.

Again conformal symmetry constrains the three point function of conserved currents and that involving conserved currents and the stress tensors to a few un-determined numbers.

Can we constrain these few undetermined numbers, and thereby constrain the allowed class of theories?

• Consider conformal field theories in $d = 3$. The 3 point functions admit a parity odd structure.

$$\langle TTT\rangle = n_s^T\langle TTT\rangle_{\rm freeboson} + n_f^T\langle TTT\rangle_{\rm freefermion} + p_T\langle TTT\rangle_{\rm parityodd}$$

$$\langle jTj\rangle = n_s^j\langle jTj\rangle_{\rm freeboson} + n_f^j\langle jTj\rangle_{\rm freefermion} + p_T^j\langle jTj\rangle_{\rm parityodd}$$

Giombi, Prakash, Yin (2011).

Compare to $d = 4$

The three point function of the stress tensor in CFT's is constrained by conformal invariance. eg. in $d = 4$

$$\langle TTT\rangle = n_s^T \langle TTT\rangle_{\rm freeboson} + n_f^T \langle TTT\rangle_{\rm freefermion} + n_v^T \langle TTT\rangle_{\rm freevector}$$

$\langle TTT\rangle_{\rm freeboson}$ is the correlator obtained by performing Wick contraction on the stress tensor of a massless free scalar theory in $d = 4$. Similar definition for the other correlators. Osborn, Petkou (1993).
n_s, n_f, n_v need not be integers are theory dependent parameters.

- The structure of the parity odd term.

$$\langle j(x)T(x_1)j(0)\rangle = \frac{\epsilon_2^\sigma I_\sigma^\alpha(x-x_1)\epsilon_3^\rho I_\rho^\beta(-x_1)\epsilon_1^{\mu\nu} t_{\mu\nu\alpha\beta}(X)}{|x_1-x|^3|x_1|^3|x|} + p_j\frac{Q_1^2 S_1 + 2P_2^2 S_3 + 2P_3^2 S_2}{|x_1-x||x||-x_1|},$$

$$
\begin{aligned}
t_{\mu\nu\alpha\beta}(X) &= \left(-\frac{2c}{3} + 2e\right) h^1_{\mu\nu}(\hat{X})\eta_{\alpha\beta} + (3e) h^1_{\mu\nu}(\hat{X}) h^1_{\alpha\beta} \\
&\quad + c h^2_{\mu\nu\alpha\beta}(\hat{X}) + e h^3_{\mu\nu\alpha\beta}, \\
Q_1^2 &= \epsilon_1^\mu \epsilon_1^\nu \left(\frac{x_{1\mu}}{x_1^2} - \frac{x_{1\mu} - x_\mu}{(x_1 - x)^2} \right) \left(\frac{x_{1\nu}}{x_1^2} - \frac{x_{1\nu} - x_\nu}{(x_1 - x)^2} \right), \\
P_2^2 &= -\frac{\epsilon_1^\mu \epsilon_1^\nu I_{\mu\nu}(x_1)}{2x_1^2}, \\
P_3^2 &= -\frac{\epsilon_1^\mu \epsilon_2^\nu I_{\mu\nu}(x_1 - x)}{2(x_1 - x)^2},
\end{aligned}
$$

$$
\begin{aligned}
S_1 &= \frac{1}{4|x_1 - x||x^3| - x_1|}\left(\varepsilon^{\mu\nu}{}_{\rho} x_\mu (x_1 - x)_\nu \epsilon_2^\rho \epsilon_3^\alpha x_\alpha\right. \\
&\quad \left. - \frac{\varepsilon^{\mu}{}_{\nu\rho}}{2}\left(|x_1 - x|^2 x_\mu + |x|^2 (x_1 - x)_\mu\right) \epsilon_2^\nu \epsilon_3^\rho\right), \\
S_2 &= \frac{1}{4|x_1 - x||x|| - x_1|^3}\left(\varepsilon^{\mu\nu}{}_{\rho} (x_{1\mu}) x_\nu \epsilon_3^\rho \epsilon_1^\alpha x_{1\alpha}\right. \\
&\quad \left. - \frac{\varepsilon^{\mu}{}_{\nu\rho}}{2}\left(-|x|^2 x_{1\mu} + |x_1|^2 x_\mu\right) \epsilon_3^\nu \epsilon_1^\rho\right), \\
S_3 &= \frac{1}{4|x_1 - x|^3|x|| - x_1|}\left(\varepsilon^{\mu\nu}{}_{\rho} (x_1 - x)_\mu (-x_{1\nu}) \epsilon_1^\rho \epsilon_2^\alpha (x_1 - x_\alpha)\right. \\
&\quad \left. - \frac{\varepsilon^{\mu}{}_{\nu\rho}}{2}\left(|x|^2 (x - x_1)_\mu + |x - x_1|^2 (-x_\mu)\right) \epsilon_1^\nu \epsilon_2^\rho\right),
\end{aligned}
$$

where

$$
\begin{aligned}
\hat{X} &= \frac{x - x_1}{|x - x_1|^2} + \frac{x_1}{|x_1|^2}, \\
I_{\alpha\beta}(x) &= \eta_{\alpha\beta} - \frac{2x_\alpha x_\beta}{x^2}, \\
h^1_{\mu\nu}(\hat{x}) &= \frac{x_\mu x \nu}{x^2} - \frac{1}{3}\eta_{\mu\nu}, \\
h^2_{\mu\nu\sigma\rho}(\hat{x}) &= \frac{x_\mu x_\sigma}{x^2}\eta_{\nu\rho} + (\mu \leftrightarrow \nu, \rho \leftrightarrow \sigma) - \frac{4}{3}\frac{x_\mu x_\nu}{x^2}\eta_{\sigma\rho} - \frac{4}{3}\frac{x_\sigma x_\rho}{x^2}\eta_{\mu\nu} \\
&\quad + \frac{3}{16}\eta_{\mu\nu}\eta_{\sigma\rho}, \\
h^3{\mu\nu\sigma\rho} &= \eta_{\mu\sigma}\eta_{\nu\rho} + \eta_{\mu\rho}\eta_{\nu\sigma} - \frac{2}{3}\eta_{\mu\nu}\eta_{\sigma\rho}, \\
c &= \frac{3(2n^j_f + n^j_s)}{256\pi^3}, \qquad e = \frac{3n^j_s}{256\pi^3}.
\end{aligned}
$$

• The existence of the parity odd term was first confirmed in large N Chern Simons theory coupled to fundamental fermions using a one loop calculation by
Giombi, Minwalla, Prakash, Trivedi, Wadia, Yin (2011), Aharony, Gur-Ari, Yacoby (2011).

We study the parity odd structure in conformal field theories in $d = 3$.

• Constraints on the parity odd structure from the average null energy condition (ANEC).

• The parity odd term and the bootstrap equations.

• Deriving the constraints from ANEC using reflection positivity, and analyticity.

CONSTRAINTS FROM ANEC

- The conformal collider:

One considers an excitation at say the origin with energy E.
The excitation propagates outward in a spherical wave.
Keep a detector in the direction $\hat{n}$, say y direction.
Measure the integrated energy over time. eg. $d = 3$

$$\hat{E}_{\hat{n}} = \lim_{r \to \infty} r \int_{-\infty}^{\infty} dt \eta^i T_i^t(t, r\hat{n})$$

$$\langle E_{\hat{n}} \rangle = \frac{\langle 0|O^\dagger E_{\hat{n}} O|0\rangle}{\langle 0|\langle O^\dagger O|0\rangle}$$

• O creates the state we are interested in.

$O \sim T_{\mu\nu}$ or $O \sim j$.

• The requirement that the integrated energy measured by the detector is positive is equivalent to the average null energy condition (ANEC).

• Thus choosing O to be T or j results in constraints on the parameters of the three points functions $\langle TTT \rangle$ or $\langle jTj \rangle$.

The excited states are defined by

$$O_E|0\rangle = \int dtdxdye^{-iEt}O(t,x,y)|0\rangle$$

We choose

$$O(\epsilon, T) = \epsilon_{ij}T^{ij} = \epsilon \cdot T \qquad O(\epsilon, j) = \epsilon_i j^i = \epsilon \cdot j$$

By conservation laws the polarizations are purely spatial.

In $d = 4$, there are 3 independent polarizations for the stress tensor and 2 independent polarization for the current.

In $d = 3$, there are 2 independent polarization for the stress tensor and 2 for the current.

Let us consider a linear combination of polarizations

$$\sum_a \alpha^a \epsilon^{(a)}$$

Then the expectation value of the energy flux operator looks like

$$\sum_{a,b} \alpha^a \alpha^b \langle \epsilon^{(a)} \cdot O | \hat{E} | \epsilon^{(b)} \cdot O \rangle = \sum_{a,b} \alpha^a \alpha^b M_{ab}$$

$$M_{ab} = \langle \epsilon^{(a)} \cdot O | \hat{E} | \epsilon^{(b)} \cdot O \rangle$$

Thus the requirement that the ANEC is satisfied turns into a condition that the eigen values of this matrix is positive.

Let us recall the conditions obtained in $d = 4$ CFT's for the $\langle TTT \rangle$ correlator.

Since there are 3 independent polarizations, the matrix is a 3×3.

There is a simple choice of polarization for which the matrix is diagonal.

Define

$$t_2 = \frac{15(-4n_v + n_f)}{n_s + 12n_v + 3n_f}, \quad t_4 = \frac{15(n_s + 2n_v - 2n_f)}{2(n_s + 12n_v + 3n_f)}$$

.

Then the condition that the three diagonal elements are positive are

$$
\begin{aligned}
1-\frac{t_2}{3}-2\frac{t_4}{15}\geq 0, && : \mathrm{I} \\
2(1-\frac{t_2}{3}-2\frac{t_4}{15})+t_2\geq 0, && : \mathrm{II} \\
\frac{3}{2}(1-\frac{t_2}{3}-2\frac{t_4}{15})+t_2+t_4\geq 0 && : \mathrm{III}
\end{aligned}
$$

There are 2 parameters t_2, t_4. The region satisfied by the inequalities is a triangle.

All $d = 4$ CFT's which satisfy the ANEC lie in the triangle.

In fact theories which admit an Einstein gravity holgraphic dual lie at the origin.

• For $d = 3$ we have 2 independent polarisations for both charge and stress tensor excitations.

We have a 2×2 matrix for both the charge and stress tensor excitations.

- The energy matrix for charge excitations $\epsilon^x = 1, \epsilon'^y = 1$.

$$\hat{E}(j) = \begin{pmatrix} \langle 0|\mathcal{O}_E^\dagger(\epsilon; j)\mathcal{E}\mathcal{O}_E(\epsilon; j)|0\rangle & \langle 0|\mathcal{O}_E^\dagger(\epsilon; j)\mathcal{E}\mathcal{O}_E(\epsilon'; j)|0\rangle \\ \langle 0|\mathcal{O}_E^\dagger(\epsilon'; j)\mathcal{E}\mathcal{O}_E(\epsilon; j)|0\rangle & \langle 0|\mathcal{O}_E^\dagger(\epsilon'; j)\mathcal{E}\mathcal{O}_E(\epsilon'; j)|0\rangle \end{pmatrix},$$

$$\langle 0|\mathcal{O}_E^\dagger(\epsilon; j)\mathcal{E}\mathcal{O}_E(\epsilon; j)|0\rangle = \frac{1}{\langle \mathcal{O}_E(\epsilon; j)|\mathcal{O}_E(\epsilon, j)\rangle} \times$$

$$\int d^3x e^{iEt} \lim_{x_1^+ \to \infty} \frac{x_1^+}{4} \int dx_1^- \langle \epsilon \cdot j(x) T_{--}(x_1) \epsilon \cdot T(0)\rangle,$$

$$\langle 0|\mathcal{O}_E^\dagger(\epsilon; j)\mathcal{E}\mathcal{O}_E(\epsilon'; j)|0\rangle =$$

$$(\langle \mathcal{O}_E(\epsilon'; j)|\mathcal{O}_E(\epsilon,' j)\rangle \langle \mathcal{O}_E(\epsilon; j)|\mathcal{O}_E(\epsilon, j)\rangle)^{-\frac{1}{2}}$$

$$\times \int d^3x e^{iEt} \lim_{x_1^+ \to \infty} \frac{x_1^+}{4} \int dx_1^- \langle \epsilon \cdot j(x) T_{--}(x_1) \epsilon' \cdot j(0)\rangle.$$

Carrying out all the steps we obtain the energy matrix for charge excitations

$$\hat{E}(j) = \begin{pmatrix} \frac{E}{4\pi}(1 - \frac{a_2}{2}) & \frac{E}{8\pi}\alpha_j \\ \frac{E}{8\pi}\alpha_j & \frac{E}{4\pi}(1 + \frac{a_2}{2}) \end{pmatrix},$$

where

$$\begin{aligned} a_2 &= -\frac{2(n_f^j - n_s^j)}{(n_f^j + n_s^j)}, \\ \alpha_j &= \frac{4\pi^4 p_j}{(n_f^j + n_s^j)}. \end{aligned}$$

The trace of this matrix is positive. Therefore the condition that the eigen values are positive leads to

$$a_2^2 + \alpha_j^2 \leq 4.$$

This region is a disc of radius 2 centered at the origin in the a_2, α_j plane.

Evaluation of the energy matrix for the stress tensor excitations proceeds similarly.
The three point function is given by

$$\langle T(x)T(x_1)T(0)\rangle = \frac{\epsilon_1^{\mu\nu}\mathcal{I}^T_{\mu\nu,\mu'\nu'}(x)\epsilon_2^{\sigma\rho}\mathcal{I}^T_{\sigma\rho,\sigma'\rho'}(x_1)\epsilon_3^{\alpha\beta}t^{\mu'\nu'\sigma'\rho'}{}_{\alpha\beta}}{x^6 x_1^6} +$$

$$p_T\frac{(P_1^2Q_1^2 + 5P_2^2P_3^2)S_1 + (P_2^2Q_2^2 + 5P_3^2P_1^2)S_2 + (P_3^2Q_3^2 + 5P_3^2P_1^2)S}{|x - x_1||x_1|| - x|}$$

where each of the tensor structure is defined.
The calculation is more involved.

- The energy matrix for stress tensor excitations.

$\epsilon_{xy} = 1; \;\; \epsilon'_{xx} = 1, \epsilon'_{yy} = -1$

$$\hat{E}(T) = \begin{pmatrix} \langle 0|\mathcal{O}_E^\dagger(\epsilon; T)\mathcal{E}\mathcal{O}_E(\epsilon; T)|0\rangle & \langle 0|\mathcal{O}_E^\dagger(\epsilon; T)\mathcal{E}\mathcal{O}_E(\epsilon'; T)|0\rangle \\ \langle 0|\mathcal{O}_E^\dagger(\epsilon'; T)\mathcal{E}\mathcal{O}_E(\epsilon; T)|0\rangle & \langle 0|\mathcal{O}_E^\dagger(\epsilon'; T)\mathcal{E}\mathcal{O}_E(\epsilon'; T)|0\rangle \end{pmatrix},$$

$$\langle 0|\mathcal{O}_E^\dagger(\epsilon; T)\mathcal{E}\mathcal{O}_E(\epsilon; T)|0\rangle = \frac{1}{\langle \mathcal{O}_E(\epsilon; T)|\mathcal{O}_E(\epsilon, T)\rangle} \times$$
$$\int d^3x e^{iEt} \lim_{x_1^+ \to \infty} \frac{x_1^+}{4} \int dx_1^- \langle \epsilon \cdot T(x) T_{--}(x_1) \epsilon \cdot T(0)\rangle,$$
$$\langle 0|\mathcal{O}_E^\dagger(\epsilon; T)\mathcal{E}\mathcal{O}_E(\epsilon'; T)|0\rangle =$$
$$(\langle \mathcal{O}_E(\epsilon'; T)|\mathcal{O}_E(\epsilon,' T)\rangle \langle \mathcal{O}_E(\epsilon; T)|\mathcal{O}_E(\epsilon, T)\rangle)^{-\frac{1}{2}}$$
$$\times \int d^3x e^{iEt} \lim_{x_1^+ \to \infty} \frac{x_1^+}{4} \int dx_1^- \langle \epsilon \cdot T(x) T_{--}(x_1) \epsilon' \cdot T(0)\rangle.$$

The final result for the energy matrix for stress tensor excitations

$$\hat{E}(T) = \begin{pmatrix} \frac{E}{4\pi}(1 - \frac{t_4}{4}) & \frac{E}{16\pi}\alpha_T \\ \frac{E}{16\pi}\alpha_T & \frac{E}{4\pi}(1 + \frac{t_4}{4}) \end{pmatrix},$$

where,

$$\begin{aligned} t_4 &= -\frac{4(n_f^T - n_s^T)}{n_f^T + n_s^T}, \\ \alpha_T &= \frac{8\pi^4 p_T}{3(n_f^T + n_s^T)}. \end{aligned}$$

The condition that the eigen values of this matrix is positive leads to

$$t_4^2 + \alpha_T^2 \leq 16.$$

• Where does Large N Chern Simons theories lie ?

$$\begin{aligned}\langle jjT\rangle &= n_s^j\langle jjT\rangle_{\text{free boson}} + n_f^j\langle jjT\rangle_{\text{free fermion}} + p_j\langle jjT\rangle_{\text{parity odd}},\\ \langle TTT\rangle &= n_s^T\langle TTT\rangle_{\text{free boson}} + n_f^T\langle TTT\rangle_{\text{free fermion}} + p_T\langle TTT\rangle_{\text{parity odd}}\end{aligned}$$

$$n_s^T(f) = n_s^j(f) = 2N\frac{\sin\theta}{\theta}\sin^2\frac{\theta}{2}, \qquad n_f^T(f) = n_s^j(f) = 2N\frac{\sin\theta}{\theta}\cos^2\frac{\theta}{2},$$
$$p_j(f) = \alpha' N\frac{\sin^2\theta}{\theta}, \qquad p_T(f) = \alpha N\frac{\sin^2\theta}{\theta},$$

where the t 'Hooft coupling is related to θ by

$$\theta = \frac{\pi N}{\kappa}.$$

• The full dependence on the t'Hooft coupling was argued using weakly broken higher spin symmetry by Maldacena, Zhiboedov (2012).

• This was perturbatively checked to all orders in t'Hooft coupling confirmed for the large N Chern Simons theory coupled for fundamental bosons. Aharony, Gur-Ari, Yacoby (2012).

• The Maldacena, Zhiboedov analysis can in principle determine the precise normalisation once one decides on the normalisation of the tensor structure of the parity odd term.

• We fix it by first taking the normalization of the parity odd term as given by Giombi, Minwalla, Prakash, Trivedi, Wadia.

• We redo the one-loop perturbative analysis to obtain

$$\alpha' = \frac{1}{\pi^4}.$$

$$\alpha = \frac{3}{\pi^4}.$$

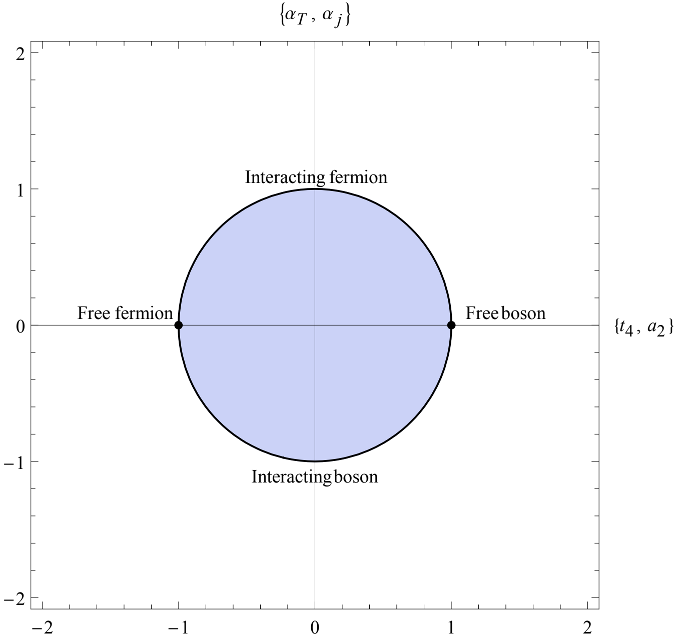
Substituting these values

$$a_2 = -2\cos\theta, \qquad \alpha_j = 2\sin\theta,$$
$$t_4 = -4\cos\theta, \qquad \alpha_T = 4\sin\theta.$$

Large N Chern-Simons theory with fundamental matter lie on the circle bounding the disc.

Their location on the bounding circle is parametrized by the 't Hooft coupling $\theta = \frac{\pi N}{\kappa}$.

To conclude



$\{\alpha_T, \alpha_j\}$
Interacting fermion
Free fermion
Free boson
$\{t_4, a_2\}$
Interacting boson
2
1
0
−1
−2
−2
−1
0
1
2

• Such conformal collider constraints on OPE coefficients were recently generalised by Córdova, Maldacena and Turiaci (1710.03199).

More importantly they also confirmed the saturation of the bounds by large N Chern-Simons theories

• The fact that Chern-Simons theories coupled to fundamental matter at large N saturate the ANEC seems to be related to the saturation of the unitarity bound for spin s currents.

$$\Delta = 1 + s + O(1/N)$$

PARITY ODD STRUCTURE

AND

BOOTSTRAP

• The parity odd structure in the 3 point functions of conserved currents introduces additional terms in the bootstrap analysis of the 4 point functions.

- Consider the 4 point function

$$\langle J(x_1)J(x_2)\phi(x_3)\phi(x_4)\rangle$$

where ϕ is a primary of dimension Δ_ϕ.

Let us define the cross ratios

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \qquad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$$

The s channel is defined by the limit $x_1 \to x_2$ In terms of cross ratios the limit is

$$u << v$$

Let us assume that the stress tensor is the lowest primary that occurs in the s-channel.

Expanding in the s channel, we have

$$
\begin{aligned}
\langle J(x_1)J(x_2)\phi(x_3)\phi(x_4)\rangle \quad = \quad & C_J \frac{H_{12}}{x_{12}^6 x_{34}^{2\Delta_\phi}} + \\
& \frac{\lambda_{\phi\phi T}}{\sqrt{C_T}} [D_{\text{even}} \mathcal{W}(2,2,\Delta_\phi,\Delta_\phi) \\
& - D_{\text{odd}} \mathcal{W}(2,2,\Delta_\phi,\Delta_\phi)]
\end{aligned}
$$

$D_{\text{even}}\mathcal{W}(2,2,\Delta_\phi,\Delta_\phi)$ is the stress tensor exchange conformal block due to the parity even terms in the $\langle JJT\rangle$ 3-point function.

It is proportional to n_s and n_f.

$D_{\text{odd}}\mathcal{W}(2,2,\Delta_\phi,\Delta_\phi)$ is the stress tensor exchange conformal block due to the parity odd terms in the $\langle JJT\rangle$ 3-point function.

It is proportional to p_j.

• How is the conformal block expansion in the s accounted for in the t channel.

The t -channel: $x_1 \to x_4$.

The contribution of the identity of the s channel can be accounted by the presence of the following composite operators in the t channel

$$
\begin{aligned}
[j,\phi]_{\tau l} &= J_\nu(\partial^{2n})\partial_{\mu_1}\partial_{\mu_2}\cdots\partial_{\mu_{l-1}}\phi, \qquad \tau_{[j,\phi]} = \Delta_\phi + 1 + 2n, \\
\widetilde{[j\phi]}_{\tau,l} &= \epsilon_{k\nu\rho}J^\nu\partial^\rho(\partial^{2n})\partial_{\mu_1}\partial_{\mu_2}\cdots\partial_{\mu_{l-1}}\phi, \qquad \tau_{\widetilde{[j,\phi]}} = \Delta_\phi + 2 + 2n.
\end{aligned}
$$

The contributions of the parity even terms in the stress tensor of the s channel gives result to the anomalous dimensions for these operators.

The analysis is performed using a formalism developed by Costa, Penedones, Poland, Rychkov (2011) to handle spinning conformal blocks.

• How are the contributions of the parity odd terms in the s channel explained.

• Examine the s channel contributions in more detail. Organise the polarizations as

$$(+) \equiv (y - t), \qquad (-) \equiv (y + t) \qquad (x)$$

Then the $(++)$ polarization: We look at

$$\langle J^+ J^+ \phi\phi \rangle$$

in the s channel.

We obtain, after removing overall scalings.

$$C_J + \frac{2\sqrt{u}}{\pi^2}(3C_J - 8\pi\lambda_{jjT})\log(v)$$

There is no contribution from the parity odd terms.

In the (xx) polarization:

$$C_J - \frac{4\sqrt{u}}{\pi^2}\left(C_J - 8\pi\lambda_{jjT}\right)\log(v)$$

The presence of the logarithms due to the stress tensor exchange.

The stress tensor spinning conformal block results from various derivatives on the following scalar block

$$G_T^{(3,3,\Delta_\phi,\Delta_\phi)}(u,v) = \frac{1}{4}\sqrt{u}(v-1)^2\, {}_2F_1\left(\frac{5}{2},\frac{5}{2},5;1-v\right)$$

In the limit $u << v << 1$, we obtain

$$\frac{1}{4}\sqrt{u}(v-1)^2\, {}_2F_1\left(\frac{5}{2},\frac{5}{2};5;1-v\right) =$$
$$\frac{32}{9\pi}\sqrt{u}(-16+6\log 4-3\log v)+O(v\sqrt{u}, v\sqrt{u}\log v)$$

The logarithmic terms contribute to the: anomalous dimensions of the composite operators in the t channel.

The parity odd terms contribute to the $(+x)$ polarization.

In the s channel we get the term

$$2p_j u^{\frac{1}{2}-\Delta_\phi} \frac{\lambda_{\phi\phi T}}{\sqrt{C_T}}$$

Note there is no identity:

The leading contribution is from the parity odd term in the stress tensor exchange.

There are no logarithms.

The reason there are no logarithms:

The parity odd spinning block for the stress tensor exchange is constructed from derivatives of the following scalar block

$$G_T^{(2,3,\Delta_\phi,\Delta_\phi)}(u,v) = \frac{4\sqrt{u}(1-v)^2}{\left(\sqrt{v}+1\right)^4\sqrt{v}}.$$

There is no logarithmic singularity:

only a square root branch cut in v.

What does crossing symmetry imply:

given that there are contribution from the parity odd terms in the s channel ?

In the t channel the contributions to the 4-point function formally are

$$\sum_{\tau,l} P_{[j\phi]_{\tau l}} \mathcal{W}_{[j\phi]_{\tau l}}(J\phi; j\phi) + \sum_{\tau,l} P_{\widetilde{[j\phi]}_{\tau l}} \mathcal{W}_{\widetilde{[j\phi]}_{\tau l}}(j\phi; j\phi)$$

$$\sum_{\tau,l} \gamma_{[j\phi]_{\tau l}} P_{[j\phi]_{\tau l}} \partial_\tau \mathcal{W}_{[j\phi]_{\tau l}}(j\phi; j\phi) + \sum_{\tau,l} \gamma_{\widetilde{[j\phi]}_{\tau l}} P_{[j\phi]_{\tau l}} \partial_\tau \mathcal{W}_{\widetilde{[j\phi]}_{\tau l}}(j\phi; j\phi)$$

$$+ \sum_{\tau,l} P_{\tilde{\mathcal{O}}_{\tau,l}} \mathcal{W}_{\tilde{\mathcal{O}}_{\tau,l}}(j\phi; j\phi)$$

$\mathcal{W}_{[j\phi]_{\tau l}}(j\phi; j\phi)$ is the conformal block for the exchange of the composite operator $[j\phi]_{\tau l}$ of spin l and twist $\tau = \Delta_\phi + 1$.
$P_{[j\phi]_{\tau l}}$ is the corresponding OPE coefficient.

$\mathcal{W}_{\widetilde{[j\phi]}_{\tau l}}(J\phi; j\phi)$ is the conformal block for the exchange of the composite operator $\widetilde{[j\phi]}_{\tau l}$ of spin l and twist $\tau = \Delta_\phi + 2$.
$P_{\widetilde{[j\phi]}_{\tau l}}$ is the corresponding OPE coefficient.

∂_τ is the derivative with respect to twist and γ the anomalous dimensions.

$\mathcal{W}_{\tilde{\mathcal{O}}_{\tau,l}}(J\phi; j\phi)$ is the parity odd conformal block for the exchange of a composite operator $\tilde{\mathcal{O}}$ of spin l and twist τ_p .
$P_{\tilde{\mathcal{O}}_{\tau,l}}$ is the corresponding OPE coefficient.

How the formal terms of the t channel expansion computed ?

In the limit $u << v << 1$ and in the large spin l approximation closed form expressions for the conformal blocks can be obtained.

This is obtained from the scalar blocks

$$G^{(\Delta_1,\Delta_2,\Delta_3,\Delta_4)}_{\mathcal{O}_{\tau,l}}(v,u) = \frac{\sqrt{l}2^{l+\tau}v^{\tau/2}u^{\frac{\Delta_1+\Delta_2-\Delta_3-\Delta_4}{4}}K_{\frac{\Delta_1+\Delta_2-\Delta_3-\Delta_4}{2}}(2l\sqrt{u})}{\sqrt{\pi}}.$$

Then the spinning blocks are obtained by appropriate derivatives on this block.

The sum over spin for the composite operators is performed by approximating the sum as an integral.

Look at the $(++)$ polarization.

• In the s channel there is a contribution from the identity and the stress tensor exchange.

• In the t channel this matched by the conformal block due to the exchange of the composite operator $[j\phi]_{\tau l}$.

• The logarithmic dependence in v in the s channel results from the anomalous dimension for the composite operator.

$$
\begin{aligned}
P_{[j,\phi]_{\tau,l}} &= \frac{\sqrt{\pi}C_J 2^{-\Delta_\phi - l+1} l^{\Delta_\phi - \frac{1}{2}}}{\Gamma(\Delta_\phi)}, \\
\gamma_{[j,\phi]_{\tau,l}} &= \frac{32\lambda_{\phi\phi T}(3C_J - 8\pi\lambda_{jjT})\Gamma(\Delta_\phi)}{3\pi^{5/2}C_J\sqrt{C_T}\Gamma(\Delta_\phi - \frac{1}{2})}
\end{aligned}
$$

There is no other conformal block which contributes in this polarization channel.

The (xx) polarization :

• The identity as well as the stress tensor exchange in the s channel is matched by both the conformal blocks due to the exchange of the composite operators $[j\phi]_{\tau l}$ as well as $\widetilde{[j\phi]}_{\tau l}$.

For the identity in the s channel the ratio of the contribution of the exchange of $[j\phi]_{\tau l}$ in the $(++)$ polarization to the (xx) polarization is

$$1 : -\frac{1}{4}$$

For the stress tensor exchange in the s channel the ratio of the contribution of the exchange of $[j\phi]_{\tau l}$ in the $(++)$ polarization to the (xx) polarization is

$$1 : -\frac{1}{3}$$

• Using the equations in this channel we can solve for the OPE coefficients and the anomalous dimensions for $\widetilde{[j,\phi]}$.

$$
\begin{aligned}
P_{\widetilde{[j,\phi]}_{\tau_0,l}} &= \frac{\sqrt{\pi}C_J 2^{-\Delta_\phi-l-1} l^{\Delta_\phi+\frac{1}{2}}}{\Gamma(\Delta_\phi)} \\
\gamma_{\widetilde{[j,\phi]}_{\tau,l}} &= \frac{64\lambda_{\phi\phi T}(16\pi\lambda_{jjT}-3C_J)\Gamma(\Delta_\phi)}{3\pi^{5/2}C_J\sqrt{C_T}\, l\, \Gamma\left(\Delta_\phi-\frac{1}{2}\right)}.
\end{aligned}
$$

• The $(--),(+-),(-+)$ polarization channels are also satisfied by these OPE coefficients and the anomalous dimensions.

• The parity odd contribution from the stress tensor exchange contributes only in the $(+x)$, $(x+)$ $(-x)$ and $(x-)$ channels.

For instance the contribution in $(+-)$ channel in the limit $u << v << 1$ is given by stress tensor exchange

$$\frac{2p_j u^{\frac{1}{2}-\Delta_\phi}\lambda_{\phi\phi T}}{\sqrt{C_T}}$$

Note there is no contribution proportional to $log(v)$.

- To reproduce the contribution s channel from the parity odd stress tensor exchange, we find that a composite operator $\tilde{O}$ of spin l and twist $\tau_p = \Delta_\phi + 1$ needs to be considered in the t channel.

The contribution in the t channel is through parity odd conformal blocks of this operator.

Summing over l and requiring the t channel to agree with the parity odd stress tensor exchange in the s channel:

The OPE coefficients for these operators are

$$P_{\tilde{\mathcal{O}}_{\tau,l}} = \frac{\sqrt{\pi}2^{-\Delta_\phi+1} p_j \lambda_{\phi\phi T}}{\Gamma\left(\Delta_\phi - \frac{1}{2}\right)} \frac{l^{(\Delta_\phi-3)}}{\sqrt{C_T}2^l}$$

$\tilde{\mathcal{O}}_{\tau l}$ is an operator of twist $\Delta_\phi + 1$ and spin l.

Therefore the composite operator $\tilde{\mathcal{O}}_{\tau l}$ is

$$[j,\phi]_{\tau l} = J_\nu \partial_{\mu_1}\partial_{\mu_2}\cdots\partial_{\mu_{l-1}}\phi, \qquad \tau_{[j,\phi]} = \Delta_\phi + 1$$

In parity odd theories, the OPE coefficient $P_{\tilde{\mathcal{O}}_{\tau,l}} \propto p_j$ is turned on.

OPE BOUNDS FROM

REFLECTION POSITIIVTY AND ANALYTICITY

• We used ANEC to derive bounds on the OPE coefficients of the three point function

$$\langle TTT \rangle \qquad \langle JJT \rangle.$$

• We will re-derive the bound on $\langle JJT \rangle$ using reflection positivity and analyticity of the conformal field theory.

• This serves as a check on the properties of the parity odd conformal blocks.

• Recall how reflection positivity and analyticity can be used to put bounds on OPE coefficients.

• Consider the following 4-point function

$$G^s(z,\bar{z}) = \frac{\langle \mathcal{O}(0)\phi(z,\bar{z})\phi(1,1)\mathcal{O}(\infty)\rangle}{\langle \phi(z,\bar{z})\phi(1,1)\rangle}$$

$\mathcal{O}, \phi$ are scalar primaries and

$$\begin{aligned} x_1 &= (0,0,0) \qquad x_2 = (\tau, y_2, 0, 0), \qquad x_3 = (0,1,0), \\ x_4 &= \lim_{a\to\infty}(0,a,0). \end{aligned}$$

Let

$$\hat{G}^s(z,\bar{z}) = G^s(e^{-2\pi i} z, \bar{z})$$

be the correlator on the second sheet.

Consider a small upper half disc around $(z, \bar{z}) = (1, 1)$ defined by

$$z = 1 + \sigma, \qquad \bar{z} = 1 + \eta\sigma,$$

σ is complex, η is real with

$$\mathrm{Im}(\sigma) \geq 0, \qquad 0 < \eta << \sigma << R << 1$$

• The correlators are now functions of σ, η . We will refer them as

$$G^s_\eta(\sigma), \qquad \hat{G}^s_\eta(\sigma)$$

Taking the $\eta \to 0$ allows us to use the closed form expressions for conformal blocks.

• Using the convergence of the s-channel, t-channel and u-channel expansions, one can show both $G^s_\eta(\sigma)$ and $\hat{G}^s_\eta(\sigma)$ are analytic in the small upper half disc.

- Using reflection positivity one can show

$$\mathrm{Re} \int_S d\sigma \left(-G^s_\eta(\sigma) + \hat{G}^s_\eta(\sigma) \right) \geq 0$$

where S is the little semi-circle bounding the upper half disc in in the counter clock wise direction.

• Using the known form of the conformal blocks one can show the leading contribution from the stress tensor in the *t*-channel

$$\left(-G^{s}_{\eta}(\sigma)+\hat{G}^{s}_{\eta}(\sigma)\right)=-iC_{\phi\phi T}C_{\mathcal{O}\mathcal{O}T}\frac{\sqrt{\eta}}{\sigma}+\cdots$$

The term proportional to $1/\sigma$ due to the presence of logarithms in the conformal block corresponding to the stress tensor exchange.

• Now using the inequality due to reflection positivity we get

$$C_{\phi\phi T}C_{\mathcal{O}\mathcal{O}T}\geq 0$$

• To derive the bounds we had using ANEC we consider the correlator

$$
\begin{aligned}
G_{\eta}^{\mu\nu}(\sigma) &= \frac{\langle J^{\mu}(0)\phi(z,\bar{z})\phi(1,1)J^{\nu}(\infty)\rangle}{\langle\phi(z,\bar{z})\phi(1,1)\rangle}, \\
\hat{G}_{\eta}^{\mu\nu}(\sigma) &= \frac{\langle J^{\mu}(0)\phi(ze^{-2\pi i},\bar{z})\phi(1,1)J^{\nu}(\infty)\rangle}{\langle\phi(z,\bar{z})\phi(1,1)\rangle}.
\end{aligned}
$$

Using analyticity and reflection positivity the following inequality is true.

$$
\begin{aligned}
\mathrm{Re}\Big[& -\int_S d\sigma \left(-G_\eta^{++}(\sigma) + \hat{G}_\eta^{++}(\sigma)\right) \\
& -\alpha \int_S d\sigma \left(-G_\eta^{+x}(\sigma) + \hat{G}_\eta^{+x}(\sigma)\right) \\
& +\alpha^* \int_S d\sigma \left(-G_\eta^{x+}(\sigma) + \hat{G}_\eta^{x+}(\sigma)\right) \\
& +\alpha\alpha^* \int_S d\sigma \left(-G_\eta^{xx}(\sigma) + \hat{G}_\eta^{xx}(\sigma)\right)\Big] \geq 0
\end{aligned}
$$

- Now minimising with respect to α we we obtain Cauchy-Schwartz inequality

$$
\mathrm{Re}\left(R^{++}R^{xx} - R^{+x}R^{x+}\right) \geq 0,
$$

$$
\begin{aligned}
R^{++} &= -\int_S d\sigma \left(-G_\eta^{++}(\sigma) + \hat{G}_\eta^{++}(\sigma)\right), \\
&= \frac{-32 C_J \sqrt{\eta} \lambda_{\phi\phi T} (a_2 + 2)}{\pi \sqrt{C_T}}, \\
R^{+x} &= -\int_S d\sigma \left(-G_\eta^{+x}(\sigma) + \hat{G}_\eta^{+x}(\sigma)\right), \\
&= \left(\frac{-\pi}{i}\right) \frac{(-8 p_j)\, \lambda_{\phi\phi T} \sqrt{\eta}}{\sqrt{C_T}}, \\
R^{x+} &= \int_S d\sigma \left(-G_\eta^{x+}(\sigma) + \hat{G}_\eta^{x+}(\sigma)\right), \\
&= \left(\frac{-\pi}{i}\right) \frac{(8 p_j)\, \lambda_{\phi\phi T} \sqrt{\eta}}{\sqrt{C_T}}, \\
R^{xx} &= \int_S d\sigma \left(-G_\eta^{xx}(\sigma) + \hat{G}_\eta^{xx}(\sigma)\right), \\
&= \frac{32 C_J \sqrt{\eta} \lambda_{\phi\phi T} (-10 + 3 a_2)}{3\pi \sqrt{C_T}}.
\end{aligned}
$$

• To obtain R^{+x} and R^{x+} we crucially used the parity odd conformal blocks in the stress channel.

This did not have logarithms but square root branch cuts.

But nevertheless the $1/\sigma$ behaviour was present $\hat{G}^{x+}_{\eta}(\sigma)$ and $\hat{G}^{xx}_{\eta}(\sigma)$.

- Substituting into the Cauchy -Schwarz inequality we obtain

$$\left(a_2 - \frac{2}{3}\right)^2 + \alpha_j^2 \leq \frac{64}{9},$$

where we have used conformal dimensions are positive which implies using a ward identity $\lambda_{\phi\phi T}$ is negative.

• However the above disc of theories we obtain is not that from ANEC.

The disc is larger and the ANEC disc is contained in it.

This is because in the (xx) polarization there is a mixture of contributions from the $++$ in the s -channel.

Recall both the composite operators $[j, \phi]_{\tau l}$ and $\widetilde{[j, \phi]}_{\tau, l}$ contributed.

• We can subtract the contributions of $[j, \phi]_{\tau I}$ and still maintain reflection positivity in R^{xx}. This results in

$$R^{xx} \to \tilde{R}^{xx} = \frac{32 C_J \sqrt{\eta} \lambda_{\phi\phi T} (a_2 - 2)}{\pi \sqrt{C_T}}$$

• Substituting back in the Cauchy-Schwarz inequality we obtain

$$a_2^2 + \alpha_j^2 \leq 4.$$

This coincides with the disc of allowed theories from ANEC.

These observations are summarized in

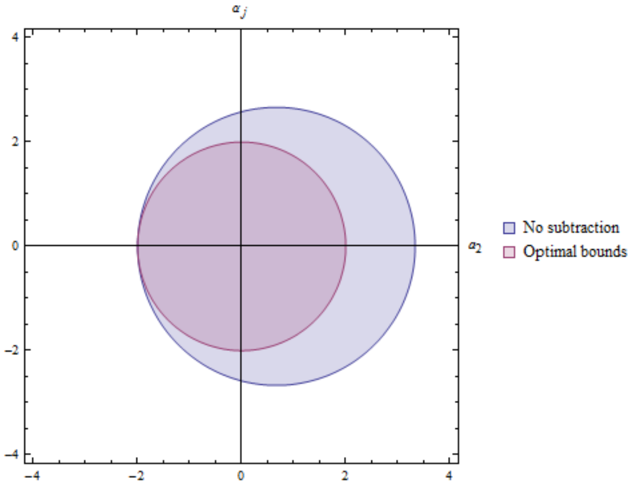


Figure: Comparative study of bounds

CONCLUSIONS

• We used ANEC to obtain bounds on the OPE coefficients involving conserved currents for parity odd conformal field theories in $d = 3$.

The theories were confined to lie inside a disc.

The boundary of the disc is populated by large N Chern Simons theories with fundamental matter.

• We showed that crossing symmetry in parity odd theories implies the existence of a new tower of composite operators.

To demonstrate this parity odd spinning conformal blocks in the stress channel were obtained.

• Reflection positivity and analyticity was used to derive the bounds obtained from ANEC.

This also served as a check on the parity odd spinning conformal blocks.