Primary Fields in Free CFT_4

Based on RdMK, P.Rabambi, R.Rabe and S.Ramgoolam, Phys. Rev. Lett. **119**, no. 16, 161602 (2017) [arXiv:1705.04039 [hep-th]], JHEP **1708**, 077 (2017) [arXiv:1705.06702 [hep-th]] and RdMK, P.Rabambi and H.J.R.Van Zyl, JHEP **1804**, 104 (2018) [arXiv:1801.10313 [hep-th]].

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Recent Trends in String Theory and Related Topics, IPM

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Motivation

CFT's defined by a list of CFT data = list of primary operators and OPE coefficients. Attempts (conformal boostrap) to constrain this data and even classify the complete set of possible CFTs. Can we list the CFT data for a free boson/fermion? Simplest example we can possibly contemplate.

Free CFT useful starting point for interacting CFT (WF fixed point). [Rychkov, Tan]

Extension to vector model will already have relevance for holographic descriptions of higher spin theories. [Polyakov, Klebanov]

Relevant to classify the possible terms on low energy effective field theory. [Henning, Lu, Melia, Murayama,]

Summary

Consider counting primaries constructed using n fields. We will obtain explicit generating functions for theories of a free boson, free bosonic vector, free bosonic matrix and a free fermion.

Introduce a polynomial representation of the conformal group and reduce the problem of constructing primaries to the problem of constructing translation invariant and harmonic polynomials.

Give explicit expressions for families of primary operators.

Counting

Free scalar is in representation V_+ ($\Delta = 1$, $(j_L, j_R) = (0,0)$) of $SO(2,4)$. Primary is $\phi(0)$. Descendants are traceless symmetric polynomials in derivatives acting on $\phi(0)$. There is a null state $\partial^2\phi(0) = 0$.

Products of n scalar fields transform in $\mathrm{Sym}^n(V_+)$ which can be decomposed into (unique) sum of irreps, each built on a primary.

To enumerate primaries constructed using n fields: (1) compute $\chi_{\mathrm{Sym}^n(R)}(M)$ and (2) decompose this into a sum of characters of irreps.

Computation of $\chi_{\mathrm{Sym}^n(R)}(M)$

Consider $M \in \mathcal{G}$ in irrep R. A useful formula: (special case of Cauchy identity)

$$\frac{1}{\det(1 - tM)} = \sum_{n=0}^{\infty} t^n \chi_{\mathrm{Sym}^n(R)}(M)$$

Apply this to $R = V_+$ with

$$M = s^D x^{J_{3,L}} y^{J_{3,R}} \in SO(2,4)$$

to obtain

$$\prod_{q=0}^{\infty} \prod_{a=-\frac{q}{2}}^{\frac{q}{2}} \prod_{b=-\frac{q}{2}}^{\frac{q}{2}} \frac{1}{1 - ts^{q+1} x^a y^b} = \sum_{n=0}^{\infty} t^n \chi_{\mathrm{Sym}^n(V_+)}(s, x, y)$$

(State $P_{\mu_1} \cdots P_{\mu_q} \phi(0)$ in V_+ has $\Delta = q + 1$ and spin $\left(\frac{q}{2}, \frac{q}{2}\right)$)

Decomposing $\mathrm{Sym}^n(V_+)$ into irreps

Unique decomposition

$$\chi_{\mathrm{Sym}^n(V_+)}(s,x,y) = \sum_{\Delta,j_1,j_2} N_{[\Delta,j_1,j_2]}\chi_{[\Delta,j_1,j_2]}(s,x,y)$$

$N_{[\Delta,j_1,j_2]}$ = number of primaries constructed from n free boson fields with dimension Δ and spin (j_1,j_2). Relevant character is [Dolan]

$$\chi_{[\Delta,j_1,j_2]}(s,x,y) = \frac{s^\Delta \chi_{j_1}(x)\chi_{j_2}(y)}{(1-s\sqrt{xy})(1-s\sqrt{\frac{x}{y}})(1-s\sqrt{\frac{y}{x}})(1-\frac{s}{\sqrt{xy}})}$$

No information in denominator - remove it by considering

$$Z_n(s,x,y) \equiv \sum_{\Delta,j_1,j_2} N_{[\Delta,j_1,j_2]} s^\Delta \chi_{j_1}(x)\chi_{j_2}(y) \qquad \text{where}$$

$$Z_n(s,x,y) = (1-s\sqrt{xy})(1-s\sqrt{\frac{x}{y}})(1-s\sqrt{\frac{y}{x}})(1-\frac{s}{\sqrt{xy}})\ \chi_{\mathrm{Sym}^n(V)}(s,x,y)$$

Generating function for the number of primaries

We have

$$Z_n(s,x,y) \equiv \sum_{\Delta,j_1,j_2} N_{[\Delta,j_1,j_2]} s^{\Delta} \chi_{j_1}(x)\chi_{j_2}(y)$$

Simplify final result using [Newton, Spradlin]

$$\begin{aligned} G_n(s,x,y) &= \sum_{d=0}^{\infty}\sum_{j_1,j_2} N_{[n+d,j_1,j_2]} s^{n+d} x^{j_1} y^{j_2} \\ &= \left[(1-\frac{1}{x})(1-\frac{1}{y}) Z_n(s,x,y)\right]_{\geq} \end{aligned}$$

Subscript $\geq$ is an instruction to keep only positive x, y powers.

$$(1-\frac{1}{x})\chi_2(x) = (1-\frac{1}{x})\left(x^2+x+1+\frac{1}{x}+\frac{1}{x^2}\right) = x^2 - \frac{1}{x^3}$$

Null States

For $n = 2$ higher spin conserved currents are primary

$$\partial_\mu J^{\mu\mu_2\cdots\mu_{2j}} = 0$$

$\Rightarrow$ null states which must be removed from character.
Null state has $\Delta = 3 + 2j$ and $j_1 = j_2 = j - \frac{1}{2}$. Subtracting null states from character we have

$$\left[\left(1-\frac{1}{x}\right)\left(1-\frac{1}{y}\right)\left(\chi_{[2+2j,j,j]}(s,x,y) - \chi_{[3+2j,j-\frac{1}{2},j-\frac{1}{2}]}(s,x,y)\right)\right]_{\geq}$$

$$= \left[\left(1-\frac{1}{x}\right)\left(1-\frac{1}{y}\right)\left(\frac{s^{2+2j}\chi_j(x)\chi_j(y) - s^{3+2j}\chi_{j-\frac{1}{2}}(x)\chi_{j-\frac{1}{2}}(y)}{(1-s\sqrt{xy})(1-s\sqrt{\frac{x}{y}})(1-s\sqrt{\frac{y}{x}})(1-\frac{s}{\sqrt{xy}})}\right)\right]_{\geq}$$

$$= \frac{s^{2+2j}x^j y^j\left(1-\frac{s}{\sqrt{xy}}\right)}{(1-s\sqrt{xy})(1-s\sqrt{\frac{x}{y}})(1-s\sqrt{\frac{y}{x}})(1-\frac{s}{\sqrt{xy}})}$$

Null States

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$$\left[\left(1-\frac{1}{x}\right)\left(1-\frac{1}{y}\right)\left(\chi_{[2+2j,j,j]}(s,x,y) - \chi_{[3+2j,j-\frac{1}{2},j-\frac{1}{2}]}(s,x,y)\right)\right]_{\geq}$$
$$= \frac{s^{2+2j}x^j y^j}{(1-s\sqrt{xy})(1-s\sqrt{\frac{x}{y}})(1-s\sqrt{\frac{y}{x}})}$$

Recall

$$Z_n(s,x,y) = (1-s\sqrt{xy})(1-s\sqrt{\frac{x}{y}})(1-s\sqrt{\frac{y}{x}})\color{red}{(1-\frac{s}{\sqrt{xy}})}\ \chi_{\mathrm{Sym}^n(V)}(s,x,y)$$

Null States

For $n = 2$ higher spin conserved currents are primary

$$\partial_\mu J^{\mu\mu_2\cdots\mu_j} = 0$$

$\Rightarrow$ null states which must be removed from character.
Null state has $\Delta = 3 + 2j$ and $j_1 = j_2 = j - \frac{1}{2}$. Subtraction achieved by removing primary that doesn't need to be subtracted dividing by $1 - s/\sqrt{xy}$ and putting original primary back in.

$$\begin{aligned} G_2(s,x,y) &= \left[(1-\frac{1}{x})(1-\frac{1}{y})\left(Z_2(s,x,y) - s^2\right)\frac{1}{1-\frac{s}{\sqrt{xy}}}\right]_{\geq} + s^2 \\ &= \sum_{j=0}^{\infty} s^{2+2j} x^j y^j \end{aligned}$$

reproducing Flato-Fronsdal Theorem.

Leading Twist Primaries

$[\Delta, j_1, j_2] = [n+q, \frac{q}{2}, \frac{q}{2}]$; each primary is a spin multiplet of $(q+1)^2$ operators. Choose state with highest spin. Corresponding generating function is $G_n^{\max}(s,x,y)$. Modify previous results in three ways:

$$\prod_{q=0}^{\infty} \frac{1}{1 - ts^{q+1}x^{\frac{q}{2}}y^{\frac{q}{2}}} = \sum_{n=0}^{\infty} t^n \chi_n^{\max}(s,x,y)$$

$$(1 - s\sqrt{xy})(1 - s\sqrt{\frac{x}{y}})(1 - s\sqrt{\frac{y}{x}})(1 - \frac{s}{\sqrt{xy}}) \to (1 - s\sqrt{xy})$$

$$(1 - \frac{1}{x})(1 - \frac{1}{y}) \to 1$$

Corresponds to using a single holomorphic derivative to generate primaries

$$G_n^{\max}(s,x,y) = \chi_n^{\max}(s,x,y)(1 - s\sqrt{xy})$$

Leading Twist Primaries

$[\Delta, j_1, j_2] = [n+q, \frac{q}{2}, \frac{q}{2}]$

Don't need to track the dependence on x and y since once n and the dimension of operator is specified, spins are determined.

$$\sum_{n=0}^{\infty} t^n G_n^{\max}(s) = \sum_{n=0}^{\infty} t^n (1-s)\chi_n^{\max}(s) = (1-s)\prod_{q=0}^{\infty} \frac{1}{1-ts^{q+1}}$$

$$G_n^{\max}(s) = \frac{s^n}{(1-s^2)(1-s^3)\cdots(1-s^n)}$$

Counting suggests a freely generated ring with $n-1$ generators.

Hilbert Series: Polynomials in d variables

Hilbert series are used to count the number of polynomials that can be formed from "basic letters". The letters may be organized by a grading. Consider $d = 2$ letters, given by x and y. Possible polynomials are:

$$1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3, \cdots$$

This is counted by the Hilbert series

$$\frac{1}{(1-s)^2} = 1 + 2s + 3s^2 + 4s^3 + 5s^4 + 6s^5 + O\left(s^6\right)$$

Consider $d = 2$ letters, given by the variables x and y^2. The possible polynomials are:

$$1, x, x^2, y^2, x^3, xy^2, x^4, x^2y^2, y^4, \cdots$$

This is counted by the Hilbert series

$$\frac{1}{(1-s)(1-s^2)} = 1 + s + 2s^2 + 2s^3 + 3s^4 + 3s^5 + O\left(s^6\right)$$

Leading Twist Primaries

Our formula for the counting of leading twist primaries

$$G_n^{\mathrm{max}}(s) = \frac{s^n}{(1-s^2)(1-s^3)\cdots(1-s^n)}$$

suggests a freely generated ring with $n-1$ generators.

A big goal of this work is to explain what this ring is and to show how it can be used to construct the primary operators explicitly.

Hilbert Series: Polynomials in d variables on the sphere

The ring is no longer freely generated - there is a constraint at level 2

$$(x_1)^2 + (x_2)^2 + \cdots (x_d)^2 = 1$$

This ring is the ring of spherical harmonics.

$$\frac{1-s^2}{(1-s)^2} = 1 + 2s + 2s^2 + 2s^3 + 2s^4 + 2s^5 + O\left(s^6\right)$$

$$\frac{1-s^2}{(1-s)^3} = 1 + 3s + 5s^2 + 7s^3 + 9s^4 + 11s^5 + O\left(s^6\right)$$

$$\frac{1-s^2}{(1-s)^4} = 1 + 4s + 9s^2 + 16s^3 + 25s^4 + 36s^5 + O\left(s^6\right)$$

Hilbert Series: Take away messages

$$\frac{1-s^2}{(1-s)^4} = 1 + 4s + 9s^2 + 16s^3 + 25s^4 + 36s^5 + O\left(s^6\right)$$

The denominator tells us about the letters being used to produce the polynomials in the ring.

The numerator tells us about relations between these letters.

$$G_n^{\max}(s) = \frac{s^n}{(1-s^2)(1-s^3)\cdots(1-s^n)}$$

There is a single letter of grading $2, 3, \cdots, n$.

There are no relations between these letters.

Extremal Primaries

$$\Delta = n + q \quad ; \quad J_3^L = \frac{q}{2}$$

J_3^R is not constrained. Using two holomorphic derivatives to generate primaries.

$$\prod_{q=0}^{\infty} \prod_{m=0}^{q} \frac{1}{(1 - ts^{q+1} x^{q/2} y^{m-q/2})} = \sum_{n=0}^{\infty} t^n \chi_n(s,x,y)$$

$$Z_n^{z,w}(s,x,y) = (1 - s\sqrt{xy})(1 - s\sqrt{x/y})\chi_n(s,x,y)$$

$$G_n^{z,w}(s,x,y) = \left[\left(1 - \frac{1}{y}\right) Z_n^{z,w}(s,x,y)\right]_{\geq}$$

Extremal Primaries

$$G_3^{z,w}(z,w) = \frac{s^3(1-s^{10}x^5y^3)}{(1-s^4x^2)(1-s^3\sqrt{x^3y^3})(1-s^2xy)(1-s^5x^{\frac{5}{2}}y^{\frac{3}{2}})}$$

$$G_4^{z,w}(s,x,y) = \frac{s^4R(s,x,y)}{(1-s^2xy)(1-s^3x^{\frac{3}{2}}y^{\frac{3}{2}})(1-s^4x^2y^2)(1-s^4x^2)\,(1-s^6x^3)\,(1-s^8x^4)}$$

where

$$\begin{aligned} R(s,x,y) \;=\;& 1+s^5x^{\frac{5}{2}}\big(\sqrt{y}+s^3x^{\frac{3}{2}}y+s^5x^{\frac{5}{2}}y+y^3-s^6x^3y^{\frac{5}{2}}-s^8x^4y^{\frac{5}{2}} \\ & -s^{16}x^8y^{\frac{7}{2}}-s^{11}x^{\frac{11}{2}}y^2\,(1+y)+s^7x^{\frac{7}{2}}\,(1-y^2)+s^4x^2y^{\frac{3}{2}}\,(1-y^2) \\ & +s^2x\sqrt{y}\,(1+y^2)-s^9x^{\frac{9}{2}}y\,(1+y^2) \\ & -s^{10}x^5y^{\frac{3}{2}}\,(1+y-y^2)-s\sqrt{x}\,(1-y-y^2)\,\big) \end{aligned}$$

Suggests a ring. Not freely generated.

Primary problem

$SO(2,4)$ algebra:

$$
\begin{aligned}
&[K_\mu, P_\nu] = 2M_{\mu\nu} - 2D\delta_{\mu\nu} \\
&[D, P_\mu] = P_\mu \\
&[D, K_\mu] = -K_\mu \\
&[M_{\mu\nu}, K_\alpha] = \delta_{\nu\alpha} K_\mu - \delta_{\mu\alpha} K_\nu \\
&[M_{\mu\nu}, K_\alpha] = \delta_{\nu\alpha} P_\mu - \delta_{\mu\alpha} P_\nu
\end{aligned}
$$

Primary:

$$\left[K_\mu, \mathcal{O}\right] = 0$$

$$\left[P_\mu, \left[P_\mu, \phi(0)\right]\right] = 0$$

Polynomial Representation

$$K_\mu = \frac{\partial}{\partial x_\mu} \qquad P_\mu = (x^2\partial_\mu - 2x_\mu x\cdot\partial - 2x_\mu)$$
$$D = (x\cdot\partial + 1) \qquad M_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu$$

For n fields, introduce n coordinates x^I_μ with $I = 1, 2, ..., n$

$$\begin{aligned} K_\mu &= \sum_{I=1}^{n} \frac{\partial}{\partial x^I_\mu} \\ P_\mu &= \sum_{I=1}^{n} \left(x^{I\rho} x^I_\rho \frac{\partial}{\partial x^I_\mu} - 2x^I_\mu x^I_\rho \frac{\partial}{\partial x^I_\rho} - 2x^I_\mu \right) \end{aligned}$$

[dMK, Ramgoolam]

Primary Polynomials

$$P_\mu P_\mu = x^4 \partial_\mu \partial_\mu$$

Primaries at dimension $n+k$ are degree k polynomials $\Psi(x^I_\mu)$ that obey

$$K_\mu \Psi = \sum_I \frac{\partial}{\partial x^I_\mu} \Psi = 0$$

$$\mathcal{L}_I \Psi = \sum_\mu \frac{\partial}{\partial x^I_\mu} \frac{\partial}{\partial x^I_\mu} \Psi = 0$$

$$\Psi(x^I_\mu) = \Psi(x^{\sigma(I)}_\mu)$$

Leading twist and extremal primaries correspond to holomorphic polynomials.

Polynomials to momenta

$$P_{\mu_1} \cdots P_{\mu_k} \cdot \phi(0) = (-1)^k 2^k k! (S^{(k)})^{\nu_1 \cdots \nu_k}_{\mu_1 \cdots \mu_k} x_{\nu_1} \cdots x_{\nu_k} \cdot 1$$

Introduce the complex coordinates

$$\begin{aligned} z &= x_1 + i x_2 \qquad & w &= x_3 + i x_4 \\ \bar{z} &= x_1 - i x_2 \qquad & \bar{w} &= x_3 - i x_4 \end{aligned}$$

that have $SO(4)$ quantum numbers

$$\begin{aligned} z &\leftrightarrow (\frac{1}{2}, \frac{1}{2}) \qquad & \bar{z} &\leftrightarrow (-\frac{1}{2}, -\frac{1}{2}) \\ w &\leftrightarrow (\frac{1}{2}, -\frac{1}{2}) \qquad & \bar{w} &\leftrightarrow (-\frac{1}{2}, \frac{1}{2}) \end{aligned}$$

$$P_z^k \phi \leftrightarrow (-1)^k 2^k k! z^k \qquad P_z^k P_w^l \phi \leftrightarrow (-1)^{k+l} 2^{k+l} (k+l)! z^k w^l$$

Solving translation invariance constraint

I of x^I_μ, with $1 \le I \le n$, transforms in natural representation, $V_{\rm nat}$ of S_n.

$$V_{\rm nat} = V_0 \oplus V_H$$

V_0 is 1-dimensional and corresponds to a single row Young diagram. V_H has dimension $(n-1)$ and corresponds to hook Young diagram.

$$X^\mu_0 = \frac{1}{\sqrt{n}} \sum_{I=1}^n x^\mu_I \qquad X^\mu_A = \frac{1}{\sqrt{A(A+1)}} (x^\mu_1 + x^\mu_2 + \cdots + x^\mu_A - A x^\mu_{A+1})$$

X^μ_0 spans V_0. X^μ_A for $A \in \{1, 2, \cdots, n-1\}$ spans V_H.

Large simple class of primaries = holomorphic S_n invariant polynomials in X^μ_A.

Constructing Leading Twist Primaries

Use $Z_A = X_A^1 + iX_A^2$. Polynomials of degree k in Z:

$$V_H^{\otimes k} = \bigoplus_{\Lambda_1 \vdash n, \Lambda_2 \vdash k} V_{\Lambda_1}^{(S_n)} \otimes V_{\Lambda_2}^{(S_k)} \otimes V_{\Lambda_1,\Lambda_2}^{Com(S_n \times S_k)}$$

Project to trivial of S_n. Also $\Lambda_2 =$ trivial $=$ single row.

$$Z_{SH}(q;\Lambda_1) = (1-q)\ q^{\frac{\sum_i c_i(c_i-1)}{2}} \prod_b \frac{1}{(1-q^{h_b})} = \sum_k q^k \mathrm{Dim}(V_{\Lambda_1,\Lambda_2}^{Com(S_n \times S_k)})$$

$c_i =$ length of the i'th column in Λ_1, b runs over boxes in Λ_1, h_b is the hook length. For the generating function $Z_n^{z,w}(s,x,y)$ we have

$$G_n^{\max}(s,x,y) = s^n Z_{SH}(s\sqrt{xy},\Lambda_1)$$

Constructing Leading Twist Primaries: Higher Spin Currents

For $n = 2$ fields primary polynomials are

$$(Z_1 - Z_2)^{2k}$$

Translating (these vanish if s is odd)

$$\begin{aligned} O_s &= (Z_1 - Z_2)^s \\ &\leftrightarrow \frac{s!}{2^s} \sum_{k=0}^{s} \frac{(-1)^k}{(k!(s-k)!)^2} \, \partial_z^{s-k}\phi \, \partial_z^k \phi \end{aligned}$$

reproducing the higher spin currents of [Craigie, Dobrev, Todorov].

Constructing Leading Twist Primaries

Consider $n = 3$ for example. Hook variables are

$$X_1^\mu = \frac{1}{\sqrt{2}}(x_1^\mu - x_2^\mu) \qquad\qquad X_2^\mu = \frac{1}{\sqrt{6}}(x_1^\mu + x_2^\mu - 2x_3^\mu)$$

Use $Z_A = X_A^1 + iX_A^2$. Rep is given by

$$\Gamma_{\yng(2,1)}(\,(12)\,) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \Gamma_{\yng(2,1)}(\,(23)\,) = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

To act on $Z_{A_1} Z_{A_2} \cdots Z_{A_k}$ use $\Gamma_k(\sigma) = \Gamma_{\yng(2,1)}(\sigma) \times \cdots \times \Gamma_{\yng(2,1)}(\sigma)$ Projector onto the trivial irrep $P_{\yng(3)} = \frac{1}{3!}\sum_{\sigma \in S_3} \Gamma_k(\sigma)$. Primary is

$$P_{A_1 A_2 \cdots A_k} = \sum_{\sigma \in S_3} \Gamma_k(\sigma)_{A_1 A_2 \cdots A_k, B_1 B_2 \cdots B_k} Z_{B_1} Z_{B_2} \cdots Z_{B_k}$$

Constructing Leading Twist Primaries: Ring Structure

Recall the generating function

$$G_n^{\max}(s) = \frac{s^n}{(1-s^2)(1-s^3)\cdots(1-s^n)}$$

The $n-1$ generators are S_n invariant version of Hook variables.

$n=2$ is generated by $(Z_1 - Z_2)^2$.

$n=3$ is generated by $(Z_1 - Z_2)^2 + (Z_1 - Z_3)^2 + (Z_2 - Z_3)^2$ and $(Z_1 + Z_2 - 2Z_3)(Z_3 + Z_2 - 2Z_1)(Z_1 + Z_3 - 2Z_2)$.

New generator at $n=4$ is $(Z_1+Z_2+Z_3-3Z_4)(Z_3+Z_2+Z_4-3Z_1)(Z_1+Z_3+Z_4-3Z_2)(Z_1+Z_2+Z_4-3Z_3)$.

Constructing Extremal Primaries

Use $Z_A = X_A^1 + iX_A^2$ and $W_A = X_A^3 + iX_A^4$. Polynomials of degree k in Z and degree l in W:

$$V_H^{\otimes k} = \bigoplus_{\Lambda_1 \vdash n, \Lambda_2 \vdash k} V_{\Lambda_1}^{(S_n)} \otimes V_{\Lambda_2}^{(S_k)} \otimes V_{\Lambda_1,\Lambda_2}^{Com(S_n \times S_k)}$$

$$V_H^{\otimes l} = \bigoplus_{\Lambda_3 \vdash n, \Lambda_4 \vdash l} V_{\Lambda_3}^{(S_n)} \otimes V_{\Lambda_4}^{(S_l)} \otimes V_{\Lambda_3,\Lambda_4}^{Com(S_n \times S_l)}$$

Project to diagonal $S_n \Rightarrow \Lambda_1 = \Lambda_3$. Also $\Lambda_2 = \Lambda_4 =$ trivial.

$$Z_{SH}(q;\Lambda_1) = (1-q)\; q^{\frac{\sum_i c_i(c_i-1)}{2}} \prod_b \frac{1}{(1-q^{h_b})}$$

c_i = length of the i'th column in Λ_1, b runs over boxes in Λ_1, h_b is the hook length. For the generating function $Z_n^{z,w}(s,x,y)$ we have

$$Z_n^{z,w}(s,x,y) = s^n \sum_{\Lambda_1 \vdash n} Z_{SH}(s\sqrt{xy},\Lambda_1) Z_{SH}(s\sqrt{\frac{x}{y}},\Lambda_1)$$

Constructing Extremal Primaries

For $n = 3$ here is an example of a family of primary polynomials

$$\Psi_n = \Big(W_3(\bar{Z}_2 - \bar{Z}_1) + W_2(\bar{Z}_1 - \bar{Z}_3) + W_1(\bar{Z}_3 - \bar{Z}_2) \Big)^{2n}$$

The translation into the free field language is

$$\mathcal{O}^{\Delta=4n+3}_{[j_1=2n, j_2=0]} \quad =$$

$$\sum_{r=0}^{2n} \sum_{s=0}^{2n-r} \sum_{t=0}^{r} \sum_{u=0}^{s} \sum_{v=0}^{2n-r-s} \frac{(2n)!(-1)^{t+u+v}}{(r-t)!t!(s-u)!u!(2n-r-s-v)!v!} \times$$

$$\times (P_w^{2n-r-s} P_{\bar{z}}^{t+s-u} \phi)(P_w^{s} P_{\bar{z}}^{r-t+v} \phi)(P_w^{r} P_{\bar{z}}^{2n+u-r-s-v} \phi)$$

Complete Class of Primaries

Harmonic condition must be accounted for. Use Hook variables and write Harmonic conditions (one in each of the n sets of coordinates) in terms of the hook variables.

There is again a ring structure. Star product on harmonic polynomials. Associative.

Construction algorithm given.

[dMK, Ramgoolam]

Vector Model

To construct a primary, distribute derivatives among

$$\phi_{I_1}\phi_{I_1}\phi_{I_2}\phi_{I_2}\cdots\phi_{I_n}\phi_{I_n}$$

I_a are summed from 1 to N.

The symmetry is broken from S_{2n} to the wreath product $S_n[S_2] \Rightarrow$ must project $V_+^{\otimes 2n}$ to trivial of $S_n[S_2]$.

$$\chi_{\mathcal{H}_n}(s,x,y) = \frac{1}{2^n n!}\sum_{\sigma\in S_n[S_2]} \mathrm{Tr}_{V^{\otimes 2n}}(\sigma M^{\otimes 2n})$$

$$M = s^{\Delta}x^{J_3^L}y^{J_3^R}$$

Vector Model

$$\sum_{n=0}^{\infty} t^n \chi_{\mathcal{H}_n}(s,x,y) = \prod_{q=0}^{\infty} \prod_{a=-\frac{q}{2}}^{\frac{q}{2}} \prod_{b=-\frac{q}{2}}^{\frac{q}{2}} \frac{1}{\sqrt{(1-ts^{2q+2}x^{2a}y^{2b})}}$$

$$\times \prod_{q_2=0}^{\infty} \prod_{a_2=-\frac{q_2}{2}}^{\frac{q_2}{2}} \prod_{b_2=-\frac{q_2}{2}}^{\frac{q_2}{2}} \prod_{q_1=0}^{\infty} \prod_{a_1=-\frac{q_1}{2}}^{\frac{q_1}{2}} \prod_{b_1=-\frac{q_1}{2}}^{\frac{q_1}{2}} \frac{1}{\sqrt{(1-ts^{q_1+q_2+2}x^{a_1+a_2}y^{b_1+b_2})}}$$

$$\begin{aligned} Z_{2n}(s,x,y) &= \chi_{\mathcal{H}_n}(s,x,y)(1-s\sqrt{xy})(1-s\sqrt{\frac{x}{y}})(1-s\sqrt{\frac{y}{x}})(1-\frac{s}{\sqrt{xy}}) \\ &= \sum_{d=0}^{\infty}\sum_{j_1,j_2} N^{O(N)}_{[2n+d,j_1,j_2]} s^{2n+d}\chi_{j_1}(x)\chi_{j_2}(y) \end{aligned}$$

$$G^{O(N)}_{2n}(s,x,y) = \sum_{d=0}^{\infty}\sum_{j_1,j_2} N^{O(N)}_{[2n+d,j_1,j_2]} s^{2n+d}x^{j_1}y^{j_2} = \left[(1-\frac{1}{x})(1-\frac{1}{y})Z_{2n}(s,x,y)\right]_{\geq}$$

Vector Model

$$G_4^{O(N),\max}(s,x,y) = \frac{s^4(1-s^6x^3y^3)}{(1-s^2xy)^2(1-s^3x^{\frac{3}{2}}y^{\frac{3}{2}})(1-s^4x^2y^2)}$$

$$G_6^{O(N),\max}(s,x,y) =$$

$$\frac{s^6(1-s\sqrt{xy}+s^3x^{\frac{3}{2}}y^{\frac{3}{2}}-s^7x^{\frac{7}{2}}y^{\frac{7}{2}}+s^9x^{\frac{9}{2}}y^{\frac{9}{2}}-s^{10}x^5y^5)}{(1-s\sqrt{xy})(1-s^2xy)^2(1-s^3x^{\frac{3}{2}}y^{\frac{3}{2}})(1-s^4x^2y^2)(1-s^6x^3y^3)}$$

By tracing over projectors to the trivial of $S_n[S_2]$ we have

$$G_{2n}^{O(N),\max}(q) = s^{2n}\sum_{\substack{\Lambda\vdash 2n\\ \Lambda even}} Z_{SH}(q,\Lambda)$$

[Caputa, dMK, Diaz]

Matrix Model

Primaries obtained by derivatives on $(\mathrm{Tr}\phi^2)\mathrm{Tr}(\phi)$, we have an $S_2 \times S_1$ invariance $\Rightarrow$ project to trivial $(\ydiagram{2}, \ydiagram{1})$ of $S_2 \times S_1$. S_2 permutes (z_1, w_1) and (z_2, w_2). This rep is subduced by $\ydiagram{3}$ and by $\ydiagram{2,1}$.

$$
\begin{aligned}
Z^{zw}_{(\mathrm{Tr}\phi^2)\mathrm{Tr}(\phi)} &= s^3 Z_{SH}(s\sqrt{xy}, \ydiagram{3}) Z_{SH}(s\sqrt{\frac{x}{y}}, \ydiagram{3}) \\
&+ 2s^3 Z_{SH}(s\sqrt{xy}, \ydiagram{2,1}) Z_{SH}(s\sqrt{\frac{x}{y}}, \ydiagram{2,1}) \\
&+ s^3 Z_{SH}(s\sqrt{xy}, \ydiagram{1,1,1}) Z_{SH}(s\sqrt{\frac{x}{y}}, \ydiagram{1,1,1}) \\
&+ s^3 Z_{SH}(s\sqrt{xy}, \ydiagram{3}) Z_{SH}(s\sqrt{\frac{x}{y}}, \ydiagram{2,1}) \\
&+ s^3 Z_{SH}(s\sqrt{xy}, \ydiagram{2,1}) Z_{SH}(s\sqrt{\frac{x}{y}}, \ydiagram{3}) \\
&+ s^3 Z_{SH}(s\sqrt{xy}, \ydiagram{2,1}) Z_{SH}(s\sqrt{\frac{x}{y}}, \ydiagram{1,1,1}) \\
&+ s^3 Z_{SH}(s\sqrt{xy}, \ydiagram{1,1,1}) Z_{SH}(s\sqrt{\frac{x}{y}}, \ydiagram{2,1}) \\
&= \frac{s^3(1+s^2x)}{(1-s\sqrt{xy})^2(1+s\sqrt{xy})(-1+s\sqrt{xy})^2(1+s\sqrt{xy})}
\end{aligned}
$$

Matrix Model

Primaries obtained by derivatives $\mathrm{Tr}(\phi^3)$, we can take $\sigma = (123)$ which is Z_3 invariant. Need to project to the trivial of Z_3. Subduced by $\ydiagram{3}$ and by $\ydiagram{1,1,1}$ Thus

$$
\begin{aligned}
Z^{zw}_{(\mathrm{Tr}\phi^3)} &= s^3 Z_{SH}(s\sqrt{xy}, \ydiagram{3}) Z_{SH}\Big(s\sqrt{\frac{x}{y}}, \ydiagram{3}\Big) \\
&+ 2s^3 Z_{SH}(s\sqrt{xy}, \ydiagram{2,1}) Z_{SH}\Big(s\sqrt{\frac{x}{y}}, \ydiagram{2,1}\Big) \\
&+ s^3 Z_{SH}(s\sqrt{xy}, \ydiagram{1,1,1}) Z_{SH}\Big(s\sqrt{\frac{x}{y}}, \ydiagram{1,1,1}\Big) \\
&+ s^3 Z_{SH}(s\sqrt{xy}, \ydiagram{3}) Z_{SH}\Big(s\sqrt{\frac{x}{y}}, \ydiagram{1,1,1}\Big) \\
&+ s^3 Z_{SH}(s\sqrt{xy}, \ydiagram{1,1,1}) Z_{SH}\Big(s\sqrt{\frac{x}{y}}, \ydiagram{3}\Big) \\
&= \frac{s^3\left(1 + s^4x^2 - (s\sqrt{x} + s^3x^{\frac{3}{2}})(\frac{1}{\sqrt{y}} + \sqrt{y}) + s^2x(\frac{1}{y} + 3 + y)\right)}{(1 - s\sqrt{\frac{x}{y}})^2(1 - s\sqrt{xy})^2(s^2\frac{x}{y} + s\sqrt{\frac{x}{y}} + 1)(1 + s\sqrt{xy} + s^2xy)}
\end{aligned}
$$

Free Fermions

The basic formula we will use is

$$\det(1+tM)=\sum_{n=0}^{\infty} t^n \chi_{(1^n)}(M)\,,$$

Since we consider fermions we must put them into the antisymmetric 1^n representation. Simple computation shows

$$\prod_{q=0}^{\infty}\prod_{a=-\frac{q+1}{2}}^{\frac{q+1}{2}}\prod_{b=-\frac{q}{2}}^{\frac{q}{2}}(1+ts^{\frac{3}{2}+q}x^a y^b)=\sum_{n=0}^{\infty} t^n \chi_{(1^n)}(s,x,y)$$

To count the primaries we again need to decompose

$$\chi_{(1^n)}(s,x,y)=\sum_{[\Delta,j_1,j_2]} N_{[\Delta,j_1,j_2]}\chi_{[\Delta,j_1,j_2]}(s,x,y)\,.$$

Free Fermions

To count the primaries we again need to decompose

$$\chi_{(1^n)}(s,x,y) = \sum_{[\Delta,j_1,j_2]} N_{[\Delta,j_1,j_2]} \chi_{[\Delta,j_1,j_2]}(s,x,y)\,.$$

Simpler to study

$$Z_n(s,x,y) \equiv \sum_{\Delta,j_1,j_2} N_{[\Delta,j_1,j_2]} s^{\Delta} \chi_{j_1}(x)\chi_{j_2}(y)\,,$$

$$Z_n(s,x,y) = (1 - s\sqrt{xy})(1 - s\sqrt{\frac{x}{y}})(1 - s\sqrt{\frac{y}{x}})(1 - \frac{s}{\sqrt{xy}})\ \chi_{(1^n)}(s,x,y)\,.$$

and finally

$$G_n(s,x,y) \equiv \sum_{\Delta,j_1,j_2} N_{[\Delta,j_1,j_2]} s^{\Delta} x^{j_1} y^{j_2}\,.$$

Free Fermions

Leading twist

$$G_n^{\rm max}(s,x,y) = (s\sqrt{xy})^{\frac{n(n-1)}{2}} (s^{\frac{3}{2}}\sqrt{x})^n \prod_{k=2}^{n} \frac{1}{1-(s\sqrt{xy})^k}$$

Extremal primaries

$$G_3^{\rm ext}(s,x,y) = \frac{s^{\frac{13}{2}} x^{\frac{5}{2}} (1 + s\sqrt{x} y^{\frac{3}{2}})}{(1-s^4x^2)(1-s^2xy)(1-s^3x^{\frac{3}{2}}y^{\frac{3}{2}})}$$

$$\begin{aligned} Z_3^{\rm ext}(s,x,y) &= s^{\frac{9}{2}} x^{\frac{3}{2}} \Big(Z_{SH}(s\sqrt{xy}, \ydiagram{3}) Z_{SH}(s\sqrt{\frac{x}{y}}, \ydiagram{1,1,1}) \\ &+ Z_{SH}(s\sqrt{xy}, \ydiagram{2,1}) Z_{SH}(s\sqrt{\frac{x}{y}}, \ydiagram{2,1}) \\ &+ Z_{SH}(s\sqrt{xy}, \ydiagram{1,1,1}) Z_{SH}(s\sqrt{\frac{x}{y}}, \ydiagram{3}) \Big) \end{aligned}$$

Free Fermions

$$K_\mu = \frac{\partial}{\partial x^\mu} \qquad D = \left(x \cdot \frac{\partial}{\partial x} - \frac{3}{2}\right)$$

$$M_{\mu\nu} = x_\mu \frac{\partial}{\partial x^\nu} - x_\nu \frac{\partial}{\partial x^\mu} + \mathcal{M}_{\mu\nu}$$

$$P_\mu = (x^2 \frac{\partial}{\partial x^\mu} - 2x_\mu x \cdot \frac{\partial}{\partial x} + 3x_\mu - 2x^\nu \mathcal{M}_{\mu\nu})$$

where

$$\mathcal{M}^{\mu\nu} = \sigma^{\mu\nu} \qquad (\sigma^{\mu\nu})_\alpha{}^\beta = \frac{1}{4}\left(\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu\right)_\alpha{}^\beta ,$$

Primary conditions:

$$\left[K_\mu, \mathcal{O}\right] = 0 = \left[\sigma^\mu P_\mu, \mathcal{O}\right]$$

Holomorphic translation invariant polynomials are a simple class.

Free Fermions

$$(z_1 - z_2)^{2s+1} \qquad \leftrightarrow$$

$$\sum_{k=0}^{2s+1} \frac{(-1)^k}{((2s-k+1)!k!)^2} (\partial_1 + i\partial_2)^k \psi(x)(\partial_1 + i\partial_2)^{2s-k+1}\psi(x)$$

[Craigie, Dobrev, Todorov]

Two more

$$\psi(0)P_z\psi(0)P_w\psi(0)$$

$$\frac{1}{3}P_w P_z^2\psi(0)P_w\psi(0)\psi(0) + \frac{1}{3}P_z\psi(0)P_w^2 P_z\psi(0)\psi(0)$$
$$+\frac{1}{4}P_w^2\psi(0)P_z^2\psi(0)\psi(0) + 2P_w P_z\psi(0)P_z\psi(0)P_w\psi(0)$$

Future directions

Have displayed a new algebraic structure: primaries of free field theory close a ring. Product of ring is NOT usual OPE. Coherence of products? Will we learn something about OPE coefficients?

Write down the primaries for small numbers of fields for free matrix and vector models.

Extension to free Dirac fermions? Free gauge fields?

Interacting CFT?

Applications to effective field theory?

Thanks for your attention!