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Finite temperature field theory

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Outline

- Review of quantum statistical mechanics
- Introducing imaginary time formalism
- Free fields at finite temperature
  - Bosonic
  - Fermionic
- Interacting fields at finite temperature
  - Finite temperature Feynman rules
  - Sample 1-loop calculation

Review of quantum statistical mechanics

Review of quantum statistical mechanics

$$\hat{\rho} = \exp\left[-\beta\left(H - \mu_i \hat{N}_i\right)\right] \qquad T^{-1} = \beta$$

$$Z = Z(V, T, \mu_1, \mu_2, \ldots) = \mathrm{Tr}\,\hat{\rho}$$

$$P = \frac{\partial(T \ln Z)}{\partial V}$$

$$N_i = \frac{\partial(T \ln Z)}{\partial \mu_i}$$

$$S = \frac{\partial(T \ln Z)}{\partial T}$$

$$E = -PV + TS + \mu_i N_i$$

Bosonic Case

$$H = \tfrac{1}{2}\omega\left(aa^\dagger + a^\dagger a\right) = \omega\left(a^\dagger a + \tfrac{1}{2}\right) = \omega\left(\hat{N} + \tfrac{1}{2}\right)$$

$$\begin{aligned} Z &= \mathrm{Tr}\, \mathrm{e}^{-\beta(H-\mu\hat{N})} = \mathrm{Tr}\, \mathrm{e}^{-\beta(\omega-\mu)\hat{N}} \\ &= \sum_{n=0}^{\infty} \langle n|\mathrm{e}^{-\beta(\omega-\mu)\hat{N}}|n\rangle = \sum_{n=0}^{\infty} \mathrm{e}^{-\beta(\omega-\mu)n} \\ &= \frac{1}{1-\mathrm{e}^{-\beta(\omega-\mu)}} \end{aligned}$$

$$N = \frac{1}{\mathrm{e}^{\beta(\omega-\mu)} - 1} \xrightarrow{\text{classical limit}} N = \mathrm{e}^{-\beta(\omega-\mu)}$$

Fermionic Case

$$\alpha^\dagger|0\rangle = |1\rangle \qquad \alpha^\dagger|1\rangle = 0$$
$$\alpha|1\rangle = |0\rangle \qquad \alpha|0\rangle = 0$$

$$\{\alpha, \alpha^\dagger\} = \alpha\alpha^\dagger + \alpha^\dagger\alpha = 1$$

$$H = \tfrac{1}{2}\omega\left(\alpha^\dagger\alpha - \alpha\alpha^\dagger\right) = \omega\left(\hat{N} - \frac{1}{2}\right)$$

$$Z = 1 + \mathrm{e}^{-\beta(\omega-\mu)}$$

$$N = \frac{1}{\mathrm{e}^{\beta(\omega-\mu)} + 1} \xrightarrow{\text{c. l.}} N = \mathrm{e}^{-\beta(\omega-\mu)}$$

Nonintractiong bosonic and fermionic gases

$$Z = \prod_{\mathbf{j}} Z_{\mathbf{j}}$$

$$\ln Z = V \int \frac{d^3p}{(2\pi)^3} \ln\left(1 \pm e^{-\beta(\omega-\mu)}\right)^{\pm 1}$$

(Fermion)

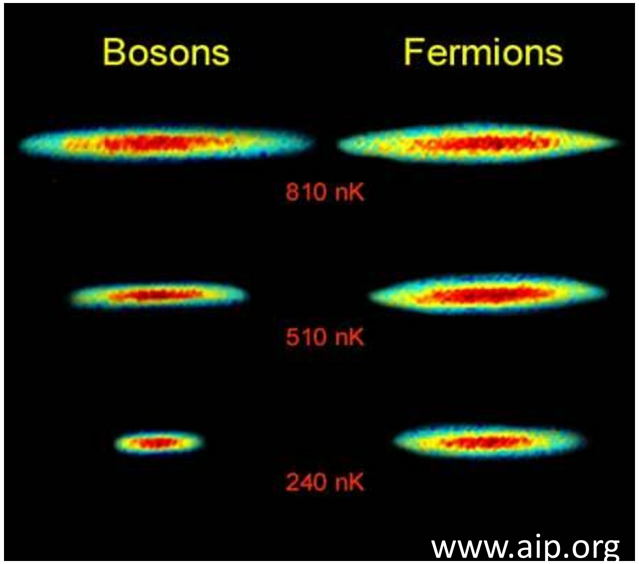
(Boson)

$$P = \frac{T}{V} \ln Z$$

$$N = V \int \frac{d^3p}{(2\pi)^3} \frac{1}{e^{\beta(\omega-\mu)} \pm 1}$$

$$E = V \int \frac{d^3p}{(2\pi)^3} \frac{\omega}{e^{\beta(\omega-\mu)} \pm 1}$$

$$P = \frac{\pi^2}{90} T^4$$ (Radiation)



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images of ultracold lithium-atom clouds

Introducing imaginary time formalism

Introducing imaginary time formalism

- Transition amplitude from a state φ_a *to a state* φ_a *after* t_f *:*

$$\langle \phi_a | \mathrm{e}^{-iHt_\mathrm{f}} | \phi_a \rangle$$

- *Compare with partition function*

$$Z = \mathrm{Tr}\, \mathrm{e}^{-\beta(H-\mu_i \hat{N}_i)} = \sum_a \int d\phi_a \, \langle \phi_a | \mathrm{e}^{-\beta(H-\mu_i \hat{N}_i)} | \phi_a \rangle$$

- Switch to an imaginary time variable τ=it

$$\Delta t = t_{\mathrm{f}}/N$$

$$\begin{aligned}
\langle\phi_a|\mathrm{e}^{-iHt_{\mathrm{f}}}|\phi_a\rangle = \lim_{N\to\infty} \int \left(\prod_{i=1}^{N} d\pi_i\, d\phi_i/2\pi\right) \\
\times \langle\phi_a|\pi_N\rangle \langle\pi_N|\mathrm{e}^{-iH\Delta t}|\phi_N\rangle \langle\phi_N|\pi_{N-1}\rangle \\
\times \langle\pi_{N-1}|\mathrm{e}^{-iH\Delta t}|\phi_{N-1}\rangle \cdots \\
\times \langle\phi_2|\pi_1\rangle \langle\pi_1|\mathrm{e}^{-iH\Delta t}|\phi_1\rangle \langle\phi_1|\phi_a\rangle
\end{aligned}$$

$$\langle\phi_{i+1}|\pi_i\rangle = \exp\left(i\int d^3x\, \pi_i(\mathbf{x})\phi_{i+1}(\mathbf{x})\right)$$

$$\langle\phi_1|\phi_a\rangle = \delta(\phi_1 - \phi_a)$$

$$
\begin{aligned}
&\langle \phi_a | \mathrm{e}^{-iHt_{\mathrm{f}}} | \phi_a \rangle \\
&\quad = \int [d\pi] \int_{\phi(\mathbf{x},0)=\phi_a(\mathbf{x})}^{\phi(\mathbf{x},t_{\mathrm{f}})=\phi_a(\mathbf{x})} [d\phi] \\
&\qquad \times \exp\left[i \int_0^{t_{\mathrm{f}}} dt \int d^3x \left(\pi(\mathbf{x},t) \frac{\partial \phi(\mathbf{x},t)}{\partial t} - \mathcal{H}\left(\pi(\mathbf{x},t), \phi(\mathbf{x},t)\right) \right) \right]
\end{aligned}
$$

$$
\left\{
\begin{array}{l}
\tau = it \\
\mathcal{H}(\pi,\phi) \to \mathcal{H}(\pi,\phi) - \mu \mathcal{N}(\pi,\phi)
\end{array}
\right.
$$

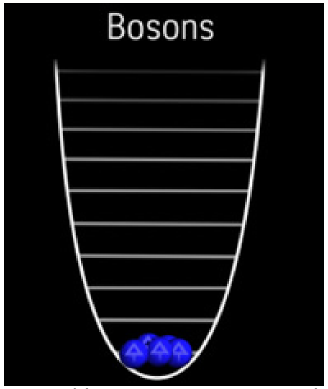
$$
\begin{aligned}
Z &= \int [d\pi] \int_{\text{periodic}} [d\phi] \\
&\quad \times \exp\left[\int_0^{\beta} d\tau \int d^3x \left(i\pi \frac{\partial \phi}{\partial \tau} - \mathcal{H}(\pi,\phi) + \mu \mathcal{N}(\pi,\phi) \right) \right]
\end{aligned}
$$

Free fields at finite temperature

Bosonic free fields

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2$$

$$\pi = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi)} = \frac{\partial\phi}{\partial t}$$



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$$\mathcal{H} = \pi\frac{\partial\phi}{\partial t} - \mathcal{L} = \tfrac{1}{2}\pi^2 + \tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2}m^2\phi^2$$

$$Z = \int_{periodic}[d\phi]\exp\left(\int_0^\beta d\tau\int d^3x\,\mathcal{L}\right)$$

Bosonic free fields

- Integrating by parts, and using the periodicity of *φ*,

$$S=-\frac{1}{2}\int_0^\beta d\tau\int d^3x\,\phi\left(-\frac{\partial^2}{\partial\tau^2}-\nabla^2+m^2\right)\phi$$

- Fourier expansion of field:

$$\phi(\mathbf{x},\tau)=\sqrt{\frac{\beta}{V}}\sum_{n=-\infty}^{\infty}\sum_{\mathbf{p}}\mathrm{e}^{i(\mathbf{p}\cdot\mathbf{x}+\omega_n\tau)}\phi_n(\mathbf{p})$$

- *Constraint of periodicity, imposes that* $\omega_n = 2\pi nT$,

$$S=-\tfrac{1}{2}\beta^2\sum_n\sum_{\mathbf{p}}(\omega_n^2+\omega^2)\phi_n(\mathbf{p})\phi_n^*(\mathbf{p})$$

$$\omega=\sqrt{\mathbf{p}^2+m^2}$$

Bosonic free fields

$$Z = N' \int [d\phi] \exp\left[-\tfrac{1}{2}(\phi, D\phi)\right] = N''(\det D)^{-1/2}$$

$$Z = \prod_n \prod_{\mathbf{p}} \left[\beta^2(\omega_n^2 + \omega^2)\right]^{-1/2}$$

$$\ln Z = -\tfrac{1}{2} \sum_n \sum_{\mathbf{p}} \ln\left[\beta^2(\omega_n^2 + \omega^2)\right]$$

$$\ln Z = V \int \frac{d^3p}{(2\pi)^3} \left[-\tfrac{1}{2}\beta\omega - \ln(1 - \mathrm{e}^{-\beta\omega})\right]$$

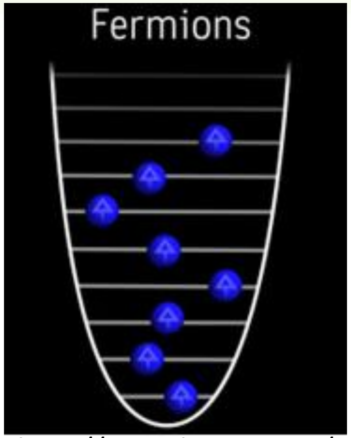
$$\left.\begin{aligned} E_0 &= -\frac{\partial}{\partial\beta} \ln Z_0 = \frac{1}{2} V \int \frac{d^3p}{(2\pi)^3}\, \omega \\ P_0 &= T \frac{\partial}{\partial V} \ln Z_0 = -\frac{E_0}{V} \end{aligned}\right\}$$

Vacuum contributions should be subtracted

Fermionic free field

$$\int d\eta_1^\dagger d\eta_1 \cdots d\eta_N^\dagger d\eta_N \, \mathrm{e}^{\eta^\dagger D \eta} = \det D$$

$$\psi_\alpha(\mathbf{x},\tau) = \frac{1}{\sqrt{V}} \sum_n \sum_{\mathbf{p}} \mathrm{e}^{i(\mathbf{p}\cdot\mathbf{x}+\omega_n\tau)} \tilde{\psi}_{\alpha;n}(\mathbf{p})$$



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$$\begin{aligned} G_{\mathrm{B}}(\mathbf{x},\mathbf{y};\tau,0) &= Z^{-1}\ \mathrm{Tr}\left[\mathrm{e}^{-\beta K}\hat{\phi}(\mathbf{x},\tau)\hat{\phi}(\mathbf{y},0)\right] \\ &= Z^{-1}\ \mathrm{Tr}\left[\hat{\phi}(\mathbf{y},0)\,\mathrm{e}^{-\beta K}\hat{\phi}(\mathbf{x},\tau)\right] \\ &= Z^{-1}\ \mathrm{Tr}\left[\mathrm{e}^{-\beta K}\mathrm{e}^{\beta K}\hat{\phi}(\mathbf{y},0)\,\mathrm{e}^{-\beta K}\hat{\phi}(\mathbf{x},\tau)\right] \\ &= Z^{-1}\ \mathrm{Tr}\left[\mathrm{e}^{-\beta K}\hat{\phi}(\mathbf{y},\beta)\hat{\phi}(\mathbf{x},\tau)\right] \\ &= Z^{-1}\ \mathrm{Tr}\left\{\hat{\rho}T_\tau\left[\hat{\phi}(\mathbf{x},\tau)\hat{\phi}(\mathbf{y},\beta)\right]\right\} \\ &= G_{\mathrm{B}}(\mathbf{x},\mathbf{y};\tau,\beta) \end{aligned}$$

$$K \equiv H - \mu\hat{Q}$$

Fermionic free field

- $$G_{\mathrm{F}}(\mathbf{x},\mathbf{y};\tau,0) = -G_{\mathrm{F}}(\mathbf{x},\mathbf{y};\tau,\beta)$$
  $$\psi(\mathbf{x},0) = -\psi(\mathbf{x},\beta)$$
  $$\omega_n = (2n+1)\pi T$$
- $$S = \sum_n \sum_{\mathbf{p}} i\tilde{\psi}^{\dagger}_{\alpha;n}(\mathbf{p}) D_{\alpha\rho} \tilde{\psi}_{\rho;n}(\mathbf{p})$$
  $$D = -i\beta\left[(-i\omega_n + \mu) - \gamma^0\gamma\cdot\mathbf{p} - m\gamma^0\right]$$
  $$Z = \det D$$

$$\ln Z = 2V\int \frac{d^3p}{(2\pi)^3}\left[\beta\omega + \ln\left(1 + \mathrm{e}^{-\beta(\omega-\mu)}\right) + \ln\left(1 + \mathrm{e}^{-\beta(\omega+\mu)}\right)\right]$$

Interacting fields at finite temperature

Perturbation expansion

$$Z = N' \int [d\phi] \mathrm{e}^{S}$$

$$S = S_0 + S_{\mathrm{I}}$$

$$\ln Z = \ln \left(N' \int [d\phi] \mathrm{e}^{S_0} \right) + \ln \left(1 + \sum_{l=1}^{\infty} \frac{1}{l!} \frac{\int [d\phi] \mathrm{e}^{S_0} S_{\mathrm{I}}^{l}}{\int [d\phi] \mathrm{e}^{S_0}} \right)$$

$$= \ln Z_0 + \ln Z_{\mathrm{I}}$$

- For $\lambda\phi^4$ theory:

$$\ln Z_1 = \frac{-\lambda \int d\tau \int d^3x \int [d\phi] \mathrm{e}^{S_0} \phi^4(\mathbf{x}, \tau)}{\int [d\phi] \mathrm{e}^{S_0}}$$

First order correction to lnZ

$$\ln Z_1 = -\lambda \int d\tau \int d^3x \sum_{n_1,\ldots,n_4} \sum_{\mathbf{p}_1,\ldots,\mathbf{p}_4} \frac{\beta^2}{V^2}$$

$$\times \exp[i(\mathbf{p}_1 + \cdots + \mathbf{p}_4) \cdot \mathbf{x}] \exp\left[i(\omega_{n_1} + \cdots + \omega_{n_4})\tau\right] \frac{A}{B}$$

$$A = \prod_l \prod_{\mathbf{q}} \int d\tilde{\phi}_l(\mathbf{q}) \exp\left[-\tfrac{1}{2}\beta^2({\omega_l}^2 + \mathbf{q}^2 + m^2)\tilde{\phi}_l(\mathbf{q})\tilde{\phi}_{-l}(-\mathbf{q})\right]$$

$$\times\ \tilde{\phi}_{n_1}(\mathbf{p}_1)\tilde{\phi}_{n_2}(\mathbf{p}_2)\tilde{\phi}_{n_3}(\mathbf{p}_3)\tilde{\phi}_{n_4}(\mathbf{p}_4)$$

$$B = \prod_l \prod_{\mathbf{q}} \int d\tilde{\phi}_l(\mathbf{q}) \exp\left[-\tfrac{1}{2}\beta^2(\omega_l^2 + \mathbf{q}^2 + m^2)\tilde{\phi}_l(\mathbf{q})\tilde{\phi}_{-l}(-\mathbf{q})\right]$$

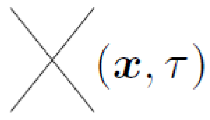
- ln Z_1 *will be zero by symmetric* integration unless $n_3 = -n_1$, $p_3 = -p_1$ *and* $n_4 = -n_2$, $p_4 = -p_2$, *or* the other two permutations.

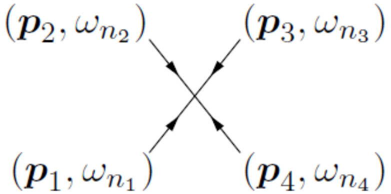
First order correction to lnZ

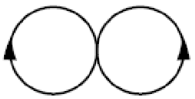
$$\frac{\int_{-\infty}^{\infty} dx\, x^2 \mathrm{e}^{-ax^2/2}}{\int_{-\infty}^{\infty} dx\, \mathrm{e}^{-ax^2/2}} = \frac{1}{a}$$

$$\ln Z_1 = -3\lambda\beta V \left(T \sum_n \int \frac{d^3p}{(2\pi)^3} \mathcal{D}_0(\omega_n, \mathbf{p}) \right)^2$$

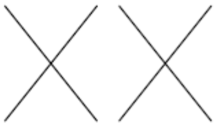
$$\mathcal{D}_0(\omega_n, \mathbf{p}) = \frac{1}{\omega_n^2 + \mathbf{p}^2 + m^2}$$

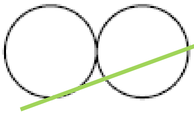
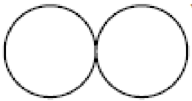


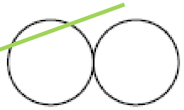
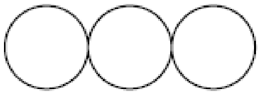


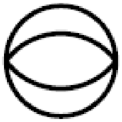
3 

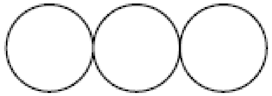
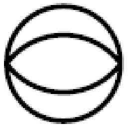
Second order correction to lnZ

$$\ln Z_2 = -\frac{1}{2}\left(\frac{\int[d\phi]\mathrm{e}^{S_0}S_{\mathrm{I}}}{\int[d\phi]\mathrm{e}^{S_0}}\right)^2 + \frac{1}{2}\frac{\int[d\phi]\mathrm{e}^{S_0}S_{\mathrm{I}}^2}{\int[d\phi]\mathrm{e}^{S_0}}$$ 

$-\frac{1}{2}(\ln Z_1)^2 = -\frac{1}{2}\Big(3$  $\otimes\ 3$  $\Big)$

$\frac{1}{2}\times 3$  $\otimes\, 3$  $+\ \frac{6\times 6\times 2}{2}$ 

$+\ \frac{4\times 3\times 2}{2}$ 

$\ln Z_2 = 36$  $+\,12$ 

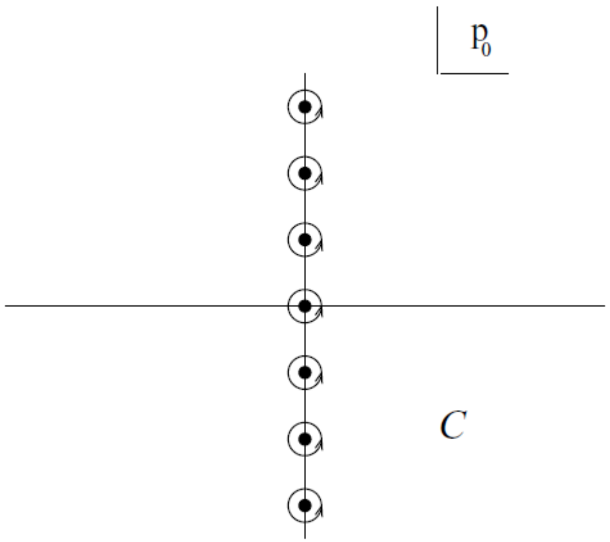
Finite temperature Feynman rules

- Draw all connected diagrams.
- Determine the combinatoric factor for each diagram.
- Include a factor $T\sum_n \int d^3p/(2\pi)^3 D_0(\omega_n, p)$ *for each line.*
- Include a factor $-\lambda$ *for each vertex.*
- Include a factor $(2\pi)^3\delta(p_{in} - p_{out})\beta\delta_{\omega_{in},\omega_{out}}$ *for each vertex, corresponding* to energy(frequency)–momentum conservation. There will be one factor $\beta(2\pi)^3\delta(0) = \beta V$ *left over.*

Sample 1-loop calculation

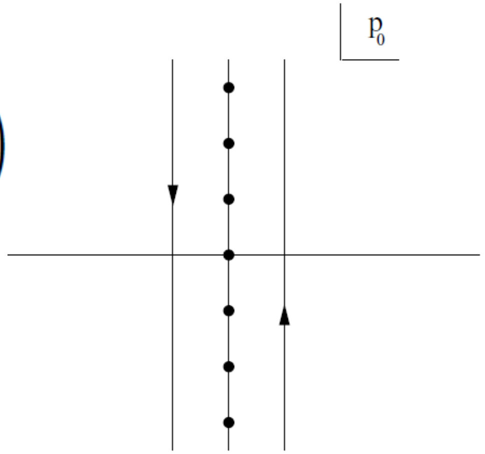
$$\ln Z_1 = -3\lambda\beta V \left(T \sum_n \int \frac{d^3p}{(2\pi)^3} \mathcal{D}_0(\omega_n, \mathbf{p}) \right)^2$$

$$T \sum_{n=-\infty}^{\infty} f(p_0 = i\omega_n = 2\pi n T i) = \frac{T}{2\pi i} \oint_c dp_0 f(p_0) \frac{1}{2} \beta \coth\left(\frac{1}{2}\beta p_0\right)$$



Sample 1-loop calculation

$$\frac{1}{2\pi i}\int_{i\infty-\epsilon}^{-i\infty-\epsilon} dp_0 f(p_0)\left(-\frac{1}{2}-\frac{1}{e^{-\beta p_0}-1}\right)$$
$$+\frac{1}{2\pi i}\int_{-i\infty+\epsilon}^{i\infty+\epsilon} dp_0 f(p_0)\left(\frac{1}{2}+\frac{1}{e^{\beta p_0}-1}\right)$$



- Setting $p_0 \rightarrow (-p_0)$ in the first integral,

$$T\sum_{n=-\infty}^{\infty} f(p_0 = i\omega_n) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dp_0\, \frac{1}{2}\left[f(p_0)+f(-p_0)\right]$$
$$+\frac{1}{2\pi i}\int_{-i\infty+\epsilon}^{i\infty+\epsilon} dp_0 \left[f(p_0)+f(-p_0)\right]\frac{1}{e^{\beta p_0}-1}$$

Sample 1-loop calculation

$$f(p_0) = -1/(p_0^2 - \omega^2)$$

$$P = \ln Z/\beta V$$

$$P_1 = -3\lambda \left(\int \frac{d^3 p}{(2\pi)^3} \frac{1}{\omega} \frac{1}{e^{\beta\omega} - 1} \right)^2$$

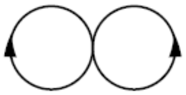
- When m = 0 and λ<<1 then,

$$P = T^4 \left(\frac{\pi^2}{90} - \frac{\lambda}{48} + \cdots \right)$$

Summary

- Imaginary time formalism ($\beta = it_f$)

$$\langle \phi_a | e^{-iHt_f} | \phi_a \rangle \qquad Z = \sum_a \int d\phi_a \, \langle \phi_a | e^{-\beta(H - \mu_i \hat{N}_i)} | \phi_a \rangle$$

- Free fields at finite temperature
- Interacting fields at finite temperature
  - Finite temperature Feynman rules $\ln Z_1 = 3$ 
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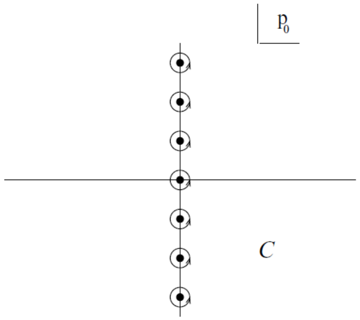
$$P = T^4 \left(\frac{\pi^2}{90} - \frac{\lambda}{48} + \cdots \right)$$



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