Problem Set No: 3

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1. The *amplification matrix* calculations:

   The amplification matrix $\mathbf{G}_{\mu\nu}$ is defined as follow,

   $$\mathbf{G}_{\mu\nu} = \frac{\partial u_{\mu}^{n+1}}{\partial u_{\nu}^{n}}.$$

   For the case of one variable the amplification matrix is reduced to amplification factor (g).

   Question: What is the amplification factor for the each case?

   (a) $u^{n+1} = u^n - f(u^n, t^n)\Delta t, \quad \Longleftrightarrow \quad$ Euler

   (b) $u^{n+1} = u^{n-1} - f(u^n, t^n) 2\Delta t, \quad \Longleftrightarrow \quad$ Leapfrog

   (c) $\begin{cases} \text{auxiliary} & u^{n+1/2} = u^n - f(u^n, t^n)\frac{\Delta t}{2}, \\ \text{main} & u^{n+1} = u^{n-1} - f(u^{n+1/2}, t^{n+1/2})\Delta t, \end{cases} \quad \Leftrightarrow \quad$ Explicit two step

   (d) $u^{n+1} = u^{n-1} - \{f(u^n, t^n) + f(u^{n+1}, t^{n+1})\}\frac{\Delta t}{2}. \quad \Leftrightarrow \quad$ Implicit two step

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2. Writ a simple code to simulate the motion of a harmonic oscillator and calculate the amplification matrix (here 2×2 matrix).

   The equation of motion for a simple oscillator is as follow,

$$\begin{aligned}\frac{dx}{dt} &= v,\\ \frac{dv}{dt} &= -x.\end{aligned}$$

   Here the time has been normalized by angular frequency of oscillation ($t \rightarrow t\omega$).

   Question: If we use the *Leapfrog* method, answer these items:

   (a) Calculate the matrix $\mathbf{G}_{\mu\nu}$ and investigate the stability condition $|g_\mu| \leq 1$.

   (b) Write a program with these initial conditions $x(0) = 1$, $v(0) = 0$ (which means the total energy of oscillation is $E = 0.5$). Do the following tasks,

      i. Plot the position as a function of time.

      ii. Plot the velocity as a function of time.

      iii. Plot velocity as a function of position.

      iv. Numerically integrate the associated phase volume and show that, it is equal to E.