Exercise 3

1. For the action

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu$$

whose solution is

$$A_\mu = \int \frac{d^3k}{(2\pi)^3\sqrt{2E_k}}\left(\epsilon_\mu(k)e^{-ikx} + \epsilon^\dagger_\mu(k)e^{ikx}\right)$$

- Work out momentum conjugate to A_μ.
- Write the equal-time commutation relations and work out the algebra of $[\epsilon_\mu(k), \epsilon^\dagger_\nu(l)]$.
- Construct the vacuum state as $\epsilon_\mu|0 >= 0$ and the Hilbert space .
- Work out $\hat{P}_\mu$ and $\hat{J}_{\mu\nu}$ in terms of ϵ and $\epsilon^\dagger$. Show that $|0>$ is Lorentz invariant.
- Compare $\hat{J}_{\mu\nu}$ with of spin-0 scalar field. Separate it into angular momentum $\hat{L}_{\mu\nu}$ and $\hat{S}_{\mu\nu}$. What is $\hat{S}_{\mu\nu}$?
- Confirm $\hat{P}_\mu\hat{S}^{\mu\nu}$=0.

2. Repeat the above problem for the case $m = 0$.
3. Show that $S^{\mu\nu}$ satisfies Lorentz algebra where $S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$.
4. Show that

- $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$.
- $\{(\gamma^\mu)^\dagger, (\gamma^\nu)^\dagger\} = 2\eta^{\mu\nu}$.
- $(S^{\mu\nu})^\dagger = \gamma^0 S^{\mu\nu}\gamma^0$.
- $U^\dagger = \gamma^0 U^{-1}\gamma^0$ where $U = e^{\frac{i}{2}w_{\mu\nu}S^{\mu\nu}}$.
- $\tilde{\gamma}^\mu = U\gamma^\mu U^{-1} = \Lambda^\mu_\nu\gamma^\nu$.
- $\{\tilde{\gamma}^\mu, \tilde{\gamma}^\nu\} = 2\eta^{\mu\nu}$.