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Final Exam of GR course

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This exam is take-home and you have four days to return the exam sheets.
Total mark: 100.

1) Linearized equations of motion on generic background $\bar{g}$.

1-1) Work out harmonic gauge in the linearized level for a generic background $\bar{g}$ (when $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$ this is called the Hilbert gauge).

1-2) How is this gauge different than the divergence-free gauge on general background?

1-3) Work out Linearized Equations Of Motion (LEOM) in the above gauge for the generic background $\bar{g}$.

1-4) Can we impose traceless, transverse (TT) gauge for a generic background $\bar{g}$?

1-5) Write the fully-gauged fixed LEOM in the TT gauge for the trace-reverse metric perturbation $\hat{h}_{\mu\nu}$ on background $\bar{g}$.

1-6) What is the "speed of Gravity Waves (GWs)"?

1-7) Assume $\bar{g}$ is a static spherically symmetric background. Write the GWs with spherical wave fronts and separate the radial and angular parts of the equations.

1-8) Let $\bar{g}$ be the AdS metric:

$$ds^2 = \frac{R^2}{z^2}\left[-dt^2 + dx^2 + dy^2 + dz^2\right], \qquad t, x, y \in \mathbb{R},\ z \in (0, \infty).$$

Write down the fully-gauge-fixed GW equation on the AdS background and analyze its dispersion replation and spectrum.

2) Kerr-Schild Coordinate system. Consider the metric of the form

$$g_{\mu\nu} = \eta_{\mu\nu} + f(x)k_\mu k_\nu$$

where k_μ is a null vector w.r.t. metric η and $f(x)$ is a scalar function.

2-1) Show that k_μ is also null w.r.t. $g_{\mu\nu}$ and that

$$g^{\mu\nu} = \eta^{\mu\nu} - f(x)k^\mu k^\nu.$$

2-2) For any *solution* to the Einstein-Maxwell theory, show that there exists a gauge in which the gauge field must necessarily be along the vector k_μ, explicitly:

$$A_\mu = A(x)k_\mu.$$

2-3) Show that Kerr-Newman geometry can be written in the Kerr-Shcild form. What are $f(x), A(x)$ and k_μ in this case?

2-4) If we denote the Killing vector field generating the horizon of the Kerr-Newman solution by ζ_H, what is the relation between ζ_H and k_μ.

3) Charged particle trajectories in the RN background. Consider a general RN solution:

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \qquad A = \frac{q}{r}dt + p\cos\theta d\phi.$$

where $f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$, $\quad Q^2 = q^2 + p^2$.

3-1) Write down the e.o.m for time-like trajectories of a particle of mass m and charge e.

3-2) Write down the null geodesic equation.

3-3) What are the quantities conserved on the above time-like or null trajectories? In particular analyze and discuss the positivity of these conserved quantities.

3-4) Analyze the circular time-like orbits.

3-5) Analyze null radial geodesics.

3-6) What are the metric and gauge field in the frame of a null radial geodesic?

3-7) Can Penrose process take place for the RN solution? If possible, what is the maximum amount of mass which one can extract from RN and how much electric charge will be necessarily extracted from the hole? (For simplicity let $p = 0$). Hint: Recall the part 3-3) of this problem.

4) Second Order Action. Consider metric perturbation,

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$$

and expand the Einstein Hilbert Lagrangian to second order in $h_{\mu\nu}$:

$$\sqrt{-g}\mathcal{R} = \sqrt{-\bar{g}}\left(\bar{\mathcal{R}} + \bar{G}_{\mu\nu}h^{\mu\nu} + \mathcal{L}_2\right) + \text{total derivative terms},$$

where $\bar{G}_{\mu\nu}$ is the Einstein tensor for the background metric $\bar{g}$.

4-1) Work out the second order Lagrangian $\mathcal{L}_2$.

4-2) Work out the total derivative terms to first order in $h_{\mu\nu}$.

4-3) Show $\mathcal{L}_2$ is invariant under gauge transformations $h_{\mu\nu} \to h_{\mu\nu} + \nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu$.

4-4) Write down the second order action $\mathcal{L}_2$ in terms of the trace-reversed perturbation

$$\hat{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}h, \quad h = \bar{g}^{\mu\nu}h_{\mu\nu}.$$

4-5) Find variation of $\mathcal{L}_2$ w.r.t. $h_{\mu\nu}$ and work out its e.o.m.

4-6) Check and compare the e.o.m. of $\mathcal{L}_2$ with the LEOM of the Einstein theory.

4-7) Compute the energy-momentum tensor for the $\mathcal{L}_2$ (by computing its variation w.r.t $\bar{g}_{\mu\nu}$). Compare this result with the Landau-Lifshitz energy-momentum pseudotensor.