Exercise 1 — Classical Bosonic String

1. The Relativistic Particle

The action describing a free relativistic point particle of mass m moving in a D-dimensional Minkowski spacetime is described by

$$S = -m \int ds = -m \int d\tau \left(-\eta_{\mu\nu} \dot{X}^\mu \dot{X}^\nu \right)^{\frac{1}{2}} , \tag{1}$$

where $\int ds$ is the length or proper time duration of the worldline traced out by the particle and $X^\mu(\tau)$ refer to the spacetime coordinates of the particle with τ any parameter that varies monotonically along the worldline. Obtain the equation of motion.

- **Reparametrization invariance.** The action (1) is manifestly invariant under the reparametrization $\tau \to \tilde{\tau}(\tau)$. Show the invariance of the action under infinitesimal reparametrization, $\tau \to \tau + \xi(\tau)$. The reparameterization invariance gives a redundancy in our description. In other words it is a *gauge symmetry* of the system.
- **Canonical analysis.** Introduce canonical conjugates and show that not all the canonical momenta are independent. Find the *primary constraint* among them. Primary constraints follow from the definition of the conjugate momenta without using any equations of motion. Show that this constraint is *first class*. A first class constraint has a *weakly* vanishing Poisson bracket with all constraints and is associated with a gauge symmetry in the system. Find the canonical Hamiltonian for the system. Count the number of physical degrees of freedom for the constrained system (1).
- **Global symmetries.** Poincaré symmetry appears as global symmetry on the worldline of the point particle (1),

$$X^\mu \to \Lambda^\mu{}_\nu X^\nu + c^\mu \tag{2}$$

  with $\Lambda^\mu{}_\nu$ and c^μ being the Lorenz transformation and constant translation parameters. Derive the corresponding Noether currents.
- **Massless particles.** The action (1) can be generalized to include the massless case by introducing an auxiliary field $e(\tau)$;

$$S = \frac{1}{2} \int d\tau \, e \left(e^{-2} \eta_{\mu\nu} \dot{X}^\mu \dot{X}^\nu - m^2 \right) . \tag{3}$$

Show that e does not introduce new degrees of freedom. Show that the constraint hold with use of equations of motion (*Secondary constraints*). Show that the action (3) is invariant under finite and infinitesimal τ-reparameterization. In the massless case derive the equations of motion and the constraint. What is the role of e in the massless case?

- **Gauge fixing.** One can think of the action (3) as if we have coupled the worldline theory to 1d gravity with e being the einbein of the metric on the worldline $e = \sqrt{-\gamma_{\tau\tau}}$. Try to use the gauge freedom in τ-reparametrization to gauge fix the e field to $1/m$ and show that this gauge identifies the τ-parameter as proper time. Use the gauge fixed action to obtain the equations of motion in this gauge. Find the geodesics describing the motion of the point particle in Minkowski spacetime. What is the answer if we have an arbitrary target space background, $\eta_{\mu\nu} \to g_{\mu\nu}$?
- **Quantization.** Study the light-cone quantization of the relativistic point particle in section 11 of Zwiebach book. Try the BRST quantization of this system as well.

2. The Relativistic String

The generalization of (1) to a one dimensional strings is to take the area of the world-sheet Σ. This is the **Nambu-Goto action**,

$$S_{NG} = -\frac{1}{2\pi\alpha'}\int_\Sigma dA = -\frac{1}{2\pi\alpha'}\int_\Sigma (-\det\gamma)^{\frac{1}{2}} \tag{4}$$

where $\gamma_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}$ is the induced metric on the world-sheet with X^μ being D scalars on the world-sheet. Work out the e.o.m and derive the primary constraint.

- **Reparametrization invariance.** Show the invariance of the action (4) under finite and infinitesimal world-sheet reparametrization.
- **Polyakov action.** Introduce the auxiliary world-sheet metric $h_{\alpha\beta}(\sigma,\tau)$ with signature $(+,-)$ to remove the square root in the Nambu-Goto action. The resulting action is called Polyakov action,

$$S_P = -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\sqrt{-h}\,h^{\alpha\beta}\gamma_{\alpha\beta}\,. \tag{5}$$

Derive the equations of motion of the Polyakov action (5) with appropriate boundary conditions.

- What are the other terms one could add to (5) (for closed sting) compatible with D-dimensional Poincaré invarinace and power counting renormalizability.
- **Energy-momentum tensor.** Define the energy-momentum tensor of the Polyakov action as the response of the system to changes in the world-sheet metric,

$$\delta S_P = \frac{1}{4\pi} \int d^2\sigma \sqrt{-h}\, T_{\alpha\beta} \delta h^{\alpha\beta} \tag{6}$$

  Find the expression for $T_{\alpha\beta}$. Using the field equations show that,

$$T_{\alpha\beta} = 0\,. \tag{7}$$

- **Local Symmetries.** Show that the Polyakov action (5) is invariant under local symmetries; reparametrization and Weyl rescaling,

$$\delta X^\mu = -\xi^\alpha \partial_\alpha X^\mu \qquad \text{and} \qquad \delta h_{\alpha\beta} = -\nabla_{(\alpha}\xi_{\beta)} + 2\Lambda h_{\alpha\beta}\,. \tag{8}$$

  Show that the energy-momentum tensor is conserved, $\nabla^\alpha T_{\alpha\beta} = 0$ (on-shell) as a consequence of diffeomorphism invariance of the action. Show that the energy-momentum tensor is traceless, $h^{\alpha\beta} T_{\alpha\beta} = 0$ as a consequence of Weyl invariance of the action.
- **Conformal gauge.** The two dimensional diffeomorphism can be used to choose coordinates such that the 2d metric is locally conformal to flat (*conformal gauge*). Show that this gauge is unique to two dimensions and as its consequence, 2d gravity is trivial (residual gauge freedom). Derive the conformal gauge preserving deffiomorphisms. When is this gauge accessible globally? (see section 5 in Polchinski I)
- **Poisson brackets.** Show that in the conformal gauge the Polyakov action simplifies to

$$S_p = \frac{1}{4\pi\alpha'} \int d^2\sigma\, (\dot{X}^2 - X'^2) = \frac{1}{\pi\alpha'} \int d^2\sigma\, \partial_+ X \partial_- X\,. \tag{9}$$

  where $\eta_{\alpha\beta} dx^\alpha dx^\beta = -d\tau^2 + d\sigma^2$ and $\sigma^\pm = \tau \pm \sigma$. In this gauge, derive the equations of motion by imposing the open/closed boundary conditions and further write the energy-momentum constraint (7). Derive the canonical momentum $\Pi^\mu = \partial\mathcal{L}/\partial \dot{X}_\mu$ and the Hamiltonian in this gauge and further by using the basic equal τ Poisson brackets,

$$\{X^\mu(\sigma), \Pi^\nu(\sigma')\} = \eta^{\mu\nu}\delta(\sigma - \sigma')\,, \tag{10}$$

obtain the generators of σ- and τ-translations.

- **Light-cone gauge.** The Light-cone gauge is a conformal gauge where the residual gauge freedom is fixed by choosing the light-cone coordinate $X^+ \propto \tau$. Show that in this gauge the energy-momentum constraints are solved explicitly and we are left only with physical degrees of freedom. Space-time light-cone coordinates are defined as $(X^\pm, X^i)$ with $i = 2, \cdots, D-1$ and $X^\pm = \frac{1}{\sqrt{2}}(X^0 \pm X^1)$.

- **Global symmetries.** The invariance of the action under spacetime global Poincaré transformations gives the Noether currents associated to translation and Lorentz symmetry. Derive these currents in conformal gauge and show their conservation on-shell. The total conserved charges are obtained by integrating the currents over a space-like section of the world-sheet $\tau = 0$.

- **Mode expansions.** Try to solve the classical equations of motion of the string in conformal gauge by showing that,

$$X^\mu(\tau, \sigma) = X_L^\mu(\sigma^+) + X_R^\mu(\sigma^-)\,. \tag{11}$$

For the closed string boundary condition $X^\mu(\tau,\sigma) = X^\mu(\tau, \sigma + 2\pi)$ show that,

$$X_R^\mu(\sigma^-) = \tfrac{1}{2}x^\mu + \tfrac{1}{2}\alpha' p^\mu \sigma^- + i\sqrt{\tfrac{\alpha'}{2}} \sum_{n\neq 0} \tfrac{1}{n}\, \alpha_n^\mu\, e^{-in\sigma^-}\,, \tag{12}$$

where $(\alpha^\mu)^\star_n = \alpha^\mu_{-n}$. Show that x^μ is the center of mass position of the string at $\tau = 0$, while p^μ is the total space-time momentum of the string. What is the expression for total angular momentum?

- **Generators of reparametrization.** From the Poisson-brackets (10) derive the brackets among the α_n^μ, $\tilde{\alpha}_n^\mu$, x^μ and p^μ. Obtain the expression of Fourier modes of constraints (7) in terms of oscillators at $\tau = 0$,

$$L_n = -\frac{1}{2\pi}\int_0^{2\pi} d\sigma\, e^{-in\sigma} T_{--} \qquad \text{and} \qquad \tilde{L}_n = -\frac{1}{2\pi}\int_0^{2\pi} d\sigma\, e^{-in\sigma} T_{++}\,. \tag{13}$$

Find the Poisson bracket algebra of L_n. Discuss that these modes are generators of $\sigma^- \to \sigma^- + e^{-in\sigma^-}$ (conformal transformation on S^1). Try to derive the effective mass of a classical string in terms of oscillator modes. Discuss why $N = \tilde{N}$ with $N = \frac{4}{\alpha'}\sum_{n>0} \alpha_n \cdot \alpha_{-n}$ and $\tilde{N} = \frac{4}{\alpha'}\sum_{n>0} \tilde{\alpha}_n \cdot \tilde{\alpha}_{-n}$?

- **Winding number.** Consider the case that we have a compact dimension, say X^{25} of radius R. Show that the spatial momentum in this direction is quantized as $p^{25} = \frac{n}{R}$ and that under $\sigma \sim \sigma + 2\pi$ X^{25} can be multi-valued,

$$X^{25}(\sigma + 2\pi) = X^{25}(\sigma) + 2\pi w R \,. \tag{14}$$

The integer w is called *winding number*. Write the mode expansion of X^{25} and try to compute the L_0 and $\tilde{L}_0$ constraints to find the effective mass of the string for an observer living in $D = 25$ non-compact directions. Show that the formula of mass is invariant under *T-dulaity*,

$$n \leftrightarrow w \qquad \text{and} \qquad R \leftrightarrow \alpha'/R \,. \tag{15}$$

- **Open string.** Find the mode expansion for the open string wave solution, when the two end points satisfy DD, NN, ND and DN boundary conditions. Show that the space-time momentum in the Dirichlet directions, carried by the open string, is not conserved. How do you understand this?

- **Regge trajectory.** Consider an open string in static gauge with $X^0 = t = \kappa\tau$, for some dimensionful constant κ. Using the constraints show that for the N boundary condiiton, the end point of the string moves at the speed of light. Consider $\kappa = L/2$ for a straight string of length $2L$ rotating at constant angular velocity in the $X^{1,2}$-plane,

$$X^1 = L\cos\left(\frac{\sigma}{2}\right)\cos\left(\frac{\tau}{2}\right) \,, \quad X^2 = L\cos\left(\frac{\sigma}{2}\right)\sin\left(\frac{\tau}{2}\right) \,, \quad X^i = 0 \;\; \text{for} \;\; i > 2 \,. \tag{16}$$

Derive the total spatial momentum P^i, energy $M = P^0$ and angular momentum $J = J_{12}$. Show that,

$$J = \alpha' M^2 \tag{17}$$

which is a straight line in the (M^2, J)-plane with slope α', called a *Regge trajectory*.