Exercise 2 — The Quantized Bosonic String

1. Quantization

We may transit from classical string to quantum mechanics of strings by improving dynamical variables to operators and replacing the Poisson brackets by commutators.

- **Light-cone quantization.** The light-cone gauge is a conformal gauge where the residual gauge freedom is fixed by choosing the light-cone coordinate $X^+ \propto \tau$. Find the proportionality constant and solve the Virasoro constraints for X^- in this gauge. Show that after this gauge fixing we are left with only $2(D-2)$ physical degrees of freedom. Space-time light-cone coordinates are defined as $(X^\pm, X^i)$ with $i = 2, \cdots, D-1$ and $X^\pm = \frac{1}{\sqrt{2}}(X^0 \pm X^1)$. Write the Polyakov action in the light-cone gauge and find the canonical Hamiltonian and identify the dynamical variables. Solve the constraints and express X^- in terms of dynamical variables.
- **Normal ordering.** Express the light-cone Hamiltonian in terms of oscillators as a normal ordered expression and includ a normal ordering constant;

$$\text{closed} \qquad H = \sum_{n>0}(\alpha^i_{-n}\alpha^i_n + \bar{\alpha}^i_{-n}\bar{\alpha}^i_n) + a + \bar{a} + \frac{\alpha'}{2}p^ip^i \tag{1}$$

$$\text{open} \qquad H = \tag{2}$$

- **Zeta-function regularization.** Determine the contribution to a from one coordinate of the closed string or the open string with DD and NN boundary condition using the zeta function regularization to regularize the sum,

$$\sum_{n\neq 0}\alpha_{-n}\alpha_n = \sum_{n\neq 0} :\alpha_{-n}\alpha_n: + \sum_{n=1}^{\infty} n\,. \tag{3}$$

For open string with ND or DN boundary conditions where the sum is on half integer modes use the generalized ζ-function to show,

$$\sum_{n=0}^{\infty}(n+\theta) = \zeta(-1,\theta) = -\frac{1}{12}(6\theta^2 - 6\theta + 1) \tag{4}$$

- **Lorenz invariance** Determine the value of a and D by demanding the commutators of the Lorenz generators in the light-cone gauge M^{-i}'s are zero.

- **Spectrum.** Construct the mass operator $m^2 = 2p^+p^- - p^ip^i$ in terms of the level number operator for the closed string and open string. Show that requiring first excited state is massless representation of $SO(D-2)$ fixes the value of a and D.

2. The Spectrum

In the quantum theory, one has to impose vanishing of Fourier components of the world-sheet energy momentum tensor as constraints on the Hilbert-space;

$$(L_n + a\,\delta_{n,0})\,|\psi\rangle = 0 \quad \text{for} \quad n \geq 0\,. \tag{5}$$

- **Old covariant quantization.** In this approach, the Hilbert space is bigger than the actual physical spectrum and is reduced to it by imposing the constraint operators on the Hilbert space. Write out the commutation relations between the Fourier modes x^μ, p^μ, α_n^μ and $\tilde{\alpha}_n^\mu$ and use them to specify the ground state and the excited states of the quantum theory for a string in momentum space. Show the presence of negative norm oscillator states.

  We introduce the normal ordering in the definition of constraints as,

$$L_n = \frac{1}{2}\sum_{m=-\infty}^{+\infty} :\alpha_{n-m}\cdot\alpha_m: \tag{6}$$

  where :: guarantees the attendance of annihilators on the right and a is due to the fact that L_0 is not completely determined by its classical expression due to ordering ambiguity. Show that $L_n^\dagger = L_{-n}$ and find out the Viraosro algebra among L_n's. Find the mass spectrum of the string,

$$M^2 = \frac{4}{\alpha'}\left(a + \sum_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n\right) = \frac{4}{\alpha'}\left(a + \sum_{n=1}^{\infty}\tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n\right)\,. \tag{7}$$

- **Physical states.** A state satisfying (5) is called *physical*. Show that a state of the form $\chi = \sum_{n=1}^{\infty} L_{-n}|\chi_n\rangle$ is orthogonal to all physical states for any $|\chi_n\rangle$. Such a state is called *spurious*. A *null* state would be both spurious and physical. Show that if $|\psi\rangle$ is physical and $|\chi\rangle$ is null then $|\psi\rangle + |\chi\rangle$ is also physical. Use the identification $|\psi\rangle \sim |\psi\rangle + |\chi\rangle$ to introduce the real physical Hilbert space as the quotient,

$$\mathcal{H}_{\mathrm{OCQ}} = \mathcal{H}_{\mathrm{phys}}/\mathcal{H}_{\mathrm{null}}\,. \tag{8}$$

- **Tachyon.** The relevant terms of Virasoro constraints are;

$$L_0 = \alpha' p^2 + \alpha_{-1} \cdot \alpha_1 + \cdots \,, \quad \text{and} \quad L_{\pm 1} = \sqrt{2\alpha'} p \cdot \alpha_{\pm 1} + \cdots \,. \tag{9}$$

  Impose the relevant physical condition (5) and show that the lowest mass level state for both open and closed string which is $|0;k\rangle$, is tachyonic.

- **Massless states.** The next mass level states of open string would be a linear combination of D states;

$$|\epsilon\,;k\rangle = \epsilon_\mu \alpha_{-1}^\mu |0;k\rangle \,. \tag{10}$$

  Show that the timelike excitation has a negative norm. Impose the physical conditions (5) at these levels and show that the physical spectrum consists of $D-2$ positive-norm states of a massless vector particle when $a=-1$. The spectrum of the closed string states is the product of two copies of the open,

$$|\epsilon\,;k\rangle = \epsilon_{\mu\nu} \alpha_{-1}^\mu \tilde{\alpha}_{-1}^\nu |0;k\rangle \,. \tag{11}$$

  Show that there are $(D-2)^2$ massless states (at $a=-1$), forming a traceless symmetric tensor, an antisymmetric tensor and a scalar. (Polchinski section 4.1).

3. State-operator Correspondence

String theory becomes a conformal field theory after fixing the conformal gauge. After continuing the signature of the closed string world-sheet metric from Minkowskian to Euclidean;

$$(\sigma^+, \sigma^-) = (\tau + \sigma, \tau - \sigma) \;\to\; -i(\tau_E + i\sigma, \tau_E - i\sigma) \equiv -i(\bar{w}, w) \tag{12}$$

with w and $\bar{w}$ being complex coordinates on the cylinder, we can map the cylinder to the complex plane via the conformal transformation $z = e^w$ and $\bar{z} = e^{\bar{w}}$ which is allowed in any conformally invariant theory. The asymptotic in- states are defined at $\tau \to -\infty$ corresponding to $z, \bar{z} \to 0$ on the z-plane. The physical string states satisfying (5) correspond to primary fields that create asymptotic states of the conformal field theory;

$$|\phi\rangle = \lim_{z,\bar{z}\to 0} \phi(z,\bar{z})|0\rangle \tag{13}$$

These fields are called *vertex operators*. In any conformal field theory there is a 1-1 correspondence between states and operators.

- **Conformal weight.** Primary fields transform as tensors of weight $(h, \bar{h})$ under conformal transformations $z \to f(z) = z + \epsilon(z)$ and $\bar{z} \to \bar{f}(\bar{z}) = \bar{z} + \bar{\epsilon}(\bar{z})$;

$$\tilde{\phi}(f, \bar{f})(df)^h (d\bar{f})^{\bar{h}} = \phi(z, \bar{z})(dz)^h (d\bar{z})^{\bar{h}} \,. \tag{14}$$

  Find the finite and infinitesimal form of the transformation. This transformation is generated by the energy-momentum tensor on plane;

$$[L_n, \phi(z)] = z^n [z\partial + (n+1)h]\phi(z) \,. \tag{15}$$

- **Normal ordered product.** $:\phi(z_1)\phi(z_2):$ is defined as the regular part of their operator product expansion (OPE);

$$\phi(z_1)\phi(z_2) = \langle \phi(z_1)\phi(z_2) \rangle + :\phi(z_1)\phi(z_2):\,, \tag{16}$$

  with $\langle \phi(z_1)\phi(z_2) \rangle$ being the two point function. Compute the OPE of the energy-momentum tensor of the world-sheet theory with itself.

- **Vertex operators.** are primary fields of the Virasoro algebra of weight $(h, \bar{h}) = (1, 1)$. The lowest closed string state is the tachyon $|0; k\rangle$ which is a space-time scalar with momentum k,

$$|0; k\rangle = \lim_{z, \bar{z} \to 0} :e^{ik \cdot X(z, \bar{z})}: |0; 0\rangle \,. \tag{17}$$

  Compute the OPE of $T(z)$ with the vertex operator and derive the conformal dimension of the vertex operator. Use the physical state condition (5) to determine the mass of the tachyon state (17). Show that;

$$p^\mu |0, k\rangle = k^\mu |0; k\rangle \,. \tag{18}$$

  What is the next level vertex operator? Derive the physical conditions by requiring this operator is primary.

4. String Scattering

Scattering amplitudes of asymptotic (physical) string states are correlation functions of the corresponding vertex operators. Show that the correlation function of two tachyon vertex operators are given by;

$$\langle :e^{ik_1 \cdot X(z, \bar{z})}::e^{ik_2 \cdot X(w, \bar{w})}: \rangle = \frac{\delta^D(k_1 + k_2)}{|z - w|^{\alpha' k_1^2}} \tag{19}$$

The vertex operators $\mathcal{O}^{(k)}(z,\bar{z})$ such as (17) creates a string state at the worldsheet location $(z,\bar{z})$. The worldsheet location is unphysical, hence we should integrate over all potential insertion points,

$$V^{(k)} = g_s \int d^2 z\, \mathcal{O}^{(k)}(z,\bar{z}) \tag{20}$$

Obviousely, only operators of weight $(1,1)$ can be integrated over and otherwise the integral must be zero or ill-defined.

Veneziano Amplitude; Consider the n-point amplitude,

$$A_n \sim \frac{1}{g_s^2}\langle V_1 \cdots V_n\rangle \sim g_s^{n-2}\int d^{2n}z\langle \mathcal{O}_1 \cdots \mathcal{O}_n\rangle \tag{21}$$

Show that the string amplitudes are $SL(2,\mathbb{C})$ invariant. Consider tachyon vertex operators and perform Wick contractions and zero mode integration to obtain,

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n\rangle \sim \delta^D(\textstyle\sum_i k_i)\prod_{i<j}|z_i - z_j|^{\alpha' k_i\cdot k_j} \tag{22}$$

Map the three points to $z = 0, \infty, 1$ by using conformal transformation and perform the following integral for $n = 4$;

$$A_4 \sim g_s^2 \delta^D(\textstyle\sum_i k_i)\int d^2 z |z-1|^{\alpha' k_3\cdot k_4}|z|^{\alpha' k_2\cdot k_4} \tag{23}$$

5. Strings in Curved Spacetime

We generalize the Polyakov action to an interacting theory describing string propagation in a curved background with metric $G_{\mu\nu}(X)$,

$$S_p = -\frac{1}{4\pi\alpha'}\int d^2\sigma \sqrt{-\gamma}\,\gamma^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu}(X) \tag{24}$$

Consider the action (24) and expand the action around the classical solution $X^\mu = x^\mu + \sqrt{\alpha'}Y^\mu$ and find the one-loop beta function associated to variation of the background metric;

$$G_{\mu\nu}(X) = G_{\mu\nu}(x) + \delta G_{\mu\nu}(x, Y)\,. \tag{25}$$

You can do this in three steps;

(a) Find the second order action $S[X] = S[x] + S^{(2)}[x,Y]$. To quadratic order in Y we then find only two terms,

$$S^{(2)} = -\frac{1}{2\pi}\int d^2\sigma\, \eta^{\alpha\beta}\left(\partial_\alpha Y^\mu \partial_\beta Y^\nu g_{\mu\nu} + \tfrac{1}{3}R_{\mu\rho\nu\sigma}\partial_\alpha x^\mu \partial_\beta x^\nu Y^\rho Y^\sigma\right) \tag{26}$$

(b) Find the one-loop renormalized couplings,

$$G_{\mu\nu}(x,\mu) = G_{\mu\nu}(x) + \alpha'\xi\, R_{\mu\nu}(x) \tag{27}$$

where ξ is the energy scale.

(c) Find the anomaly in the trace of the renormalised stress-energy tensor,

$$\eta_{\alpha\beta}T^{\alpha\beta} = -\frac{1}{2\alpha'}\beta_{\mu\nu}\partial_\alpha X^\mu \partial^\alpha X^\nu \qquad \text{with} \qquad \beta_{\mu\nu} = \frac{\partial}{\partial\xi}G_{\mu\nu}\,. \tag{28}$$

The only way to have correct degrees of freedom and make string theory consistent is to restore Weyl symmetry which amounts to set $\beta_{\mu\nu} = 0$ which yields the Einstein equation. In other words, quantum strings can propagate only on Einstein backgrounds. Try to the above analysis to 2-loop order.

If we expand the background metric around flat space as $G_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $h_{\mu\nu} \sim \epsilon_{\mu\nu}(k)e^{ik\cdot X}$, propagation in curved spacetime corresponds to coupling the string fields to the graviton sector of closed string massless excitations. We are naturally lead to include also the antisymmetic (B-filed) and the scalar (Dilaton) massless excitations. Show that the following generalization of (24) respects all target space symmetries,

$$\begin{aligned} S &= S_p + S_B + S_\Phi \qquad\qquad (29)\\ S &= \frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-\gamma}\left[\left(\gamma^{\alpha\beta}G_{\mu\nu}(X) + i\epsilon^{\alpha\beta}B_{\mu\nu}(X)\right)\partial_\alpha X^\mu \partial_\beta X^\nu + \alpha' R^{(2)}\Phi(X)\right] \end{aligned}$$

Show that the coupling of the dilaton breaks the two dimensional classical Weyl invariance while other terms preserve it. Fluctuation around classical solutions should preserve Diff×Weyl symmetry. It can be shown that at the one-loop level the energy momentum tensor is not traceless;

$$-2\alpha'\langle T_a{}^a\rangle = \alpha'\beta^\Phi R^{(2)} + \beta^G_{\mu\nu}\gamma_{\alpha\beta}\partial^\alpha X\partial^\beta X + \beta^B_{\mu\nu}\epsilon^{\alpha\beta}\partial_\alpha X\partial_\beta X\,. \tag{30}$$

The procedure for imposing Weyl invariance can be carried out by computing the β-functions and demanding them to vanish. These are the standard equations for a graviton, a two-form field and a scalar. They follow from action in spacetime,

$$S \sim \int d^D x \sqrt{-G} e^{-2\Phi} \left(R - \tfrac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} + 4\partial_\mu \Phi \partial^\mu \Phi - \frac{2(D-26)}{3\alpha'} \right) + \mathcal{O}(\alpha') \qquad (31)$$

This action describes the low-energy physics of string theory from the spacetime background point of view. Derive the field equations. The trivial solution (flat background) of these equations are $G_{\mu\nu} = \eta_{\mu\nu}$, $B_{\mu\nu} = 0$ and $\Phi = \Phi_0$.

For the open string with $a = 0, \cdots, p$ the follwoing boundary term should be added to (29),

$$\oint_{\partial\Sigma} ds \left(\partial_s X^a A_a(X) + \frac{1}{2\pi} k(s) \Phi(X) \right) \qquad (32)$$

where a background gauge field couples to the boundary of the world-sheet and the dilaton couples to the trace of the extrinsic curvature of the boundary; $k = t^\alpha t^\beta \nabla_\alpha n_\beta$ with t^α and n^α are unit vectors tangent and normal to the boundary. Show that when dilaton is zero, the quantum Weyl anomaly is described by a beta function,

$$\beta^A_a \sim \alpha'^2 \partial_a F^{ab} \,. \qquad (33)$$

Requiring that the classical coupling of A respects Weyl symmetry amounts to the Maxwell equation $\partial_a F^{ab} = 0$. The gauge field A has to leave on the Dp-brane to which the string ends are constrained. Including higher order corrections in α' leads to the exact Born-Infeld action in a flat background,

$$S \sim \int d^{p+1} x \sqrt{-\det\left(\eta_{ab} + 2\pi\alpha' F_{ab}\right)} \qquad (34)$$

If we also add the effect of all the closed string fields to the Dp-brane action we have,

$$S \sim \int d^{p+1} x \, e^{-\Phi} \sqrt{-\det\left(g_{ab} + 2\pi\alpha' F_{ab} + B_{ab}\right)} \qquad (35)$$

where g_{ab} and B_{ab} are the pull back of background metric $G_{\mu\nu}$ and the two form field $B_{\mu\nu}$ on the Dp-brane. Derive the field equations for the action (35) and show that this action is invariant under,

$$B_{ab} \to B_{ab} + \partial_{[a} \Lambda_{b]} \qquad \text{and} \qquad A_a \to A_a - \tfrac{1}{2\pi\alpha'} \Lambda_a \qquad (36)$$

The actions (31) and (35) are written in the *string frame* where the scalar degrees of freedom is mixed with the gravitational or the gauge degrees of freedom. Try to go to the *Einstein frame* where the canonical kinetic term for each field is recovered by rescaling the metric $G_{\mu\nu} \to f(\Phi) G_{\mu\nu}$, and the gauge field $A_\mu \to g(\Phi) A_\mu$.