Exercise 3 — Supersymmetry for Superstrings

1. SUSY generalities

(**suggested reading : Vafa et. al., *Mirror symmetry*, Ch 10**)

Supersymmetry often leads to: I) stability of quantum systems, and II) control over the non-perturbative regime of quantum systems.

As for item I, in string theory supersymmetry allows removing the bosonic string tachyon in Minkowski background. (Question: do you remember the masses of the open and closed bosonic string tachyons in Minkowski background?) In other words, the superstring spectrum, unlike the bosonic spectrum, does not signal a perturbative instability in the system.

The fact that supersymmetry (SUSY) removes the tachyon (after GSO, of course), is in accord with the general wisdom that supersymmetry often stabilizes quantum systems.

- Write down the $\mathcal{N} = (1,1)$ SUSY algebra (in terms of Hilbert-space operators) in 1+0, 1+1, and 1+3 dimensional quantum field theories. [Hint: ask Google.] Make sure you understand the role of †. Show that in all cases the SUSY algebra implies a positive semi-definite energy. [Hint: write the expectation value of the Hamiltonian in terms of the supercharge operators.]
- Write down the SUSY algebra in 1+1 dimensional conformal field theory (CFT). This CFT could be the one living on the worldsheet of superstrings. Does the algebra imply a positivity constraint on the spectrum?

Task: read about Witten's SUSY-inspired proof of the positive energy theorem in classical GR.

As for item II, in string theory supersymmetry allows understanding various strong-weak dualities between the five superstring theories and M-theory. In particular, a strong-weak supersymmetric duality known as the AdS/CFT correspondence has allowed a non-perturbative formulation of superstring theory on asymptotically AdS spacetimes.

The power that SUSY provides for control over non-perturbative regimes of quantum systems is demonstrated by the *Witten index* in SUSY quantum mechanics.

- In the simplified setting of 1+0 dimensional QFT (*viz* quantum mechanics), use the SUSY algebra to argue that states with non-zero energy come in bose-fermi pairs.
- Use the information from the previous problem to argue that the quantum mechanical Witten index $\mathrm{Tr}(-1)^F$ is invariant under smooth changes of couplings in SUSY quantum systems under sufficiently nice conditions.

Task: read about Witten's supersymmetric index in SUSY QFTs in various dimensions, and how it allows extracting non-perturbative information under sufficiently nice conditions. See Witten's classic paper *Constraints on supersymmetry breaking.*

Task: study the *super-particle* in Homework Problems 4.1–4.3 of Becker-Becker-Schwarz.

Task: make sure you understand the Lagrangian and the Hamiltonian formulations of supersymmetry, separately.

2. Worldsheet SUSY
(**suggested reading: GSW, and section 4 of 0201253**)

To approach the worldsheet supersymmetric conformal field theory (SCFT) of the superstring via a Lagrangian, we need to study 2d fermion fields.

- Task: read about fermions in diverse dimensions. (See section 4.1 of 0201253 for example.)
- In which dimensions can we have Majorana-Weyl fermions? How many on-shell degrees of freedom do they have? How many on-shell degrees of freedom do Majorana fermions have in 2, 4, and 10 dimensions?
- (This is a warmup problem for familiarity with 2d fermions; we won't use its result.) Given a pair of 2d Majorana fermions ψ and χ, show that

$$\psi_A\bar{\chi}_B = -\frac{1}{2}\left(\bar{\chi}\psi\delta_{AB} + \bar{\chi}\gamma_\alpha\psi(\gamma^\alpha)_{AB} + \bar{\chi}\gamma_3\psi(\gamma_3)_{AB}\right), \tag{1}$$

where $\gamma_3 = \gamma_0\gamma_1$. [Hint: one way of doing it is checking the matrix components of the two sides, using the explicit 2d gamma matrices, and the relation $\bar{\psi} := \psi^\dagger(i\gamma^0)$.]

- Derive the equations of motion for the worldsheet action (see subsection 4.6 of 0201253 for a more general action, with NS-NS background fields turned on)

$$\int \mathrm{d}^2\sigma(\partial_\alpha X_\mu \partial^\alpha X^\mu + \bar{\psi}^\mu \gamma^\alpha \partial_\alpha \psi_\mu), \tag{2}$$

  and recognize the supersymmetric partnerships in them. Make sure you are comfortable working with 2d Majorana spinors ψ, as well as 2d Majorana-Weyl spinors ψ_-, ψ_+, especially in analyzing the above action and its resulting equations of motion.

- Show that the above action is supersymmetric on shell. Then use Noether's procedure for supersymmetry to derive the conserved supercurrents.

- In order to reduce the computational load of checking supersymmetry in Lagrangians, one often works in the "superspace", parameterized by (σ^α, θ). On superspace live "superfields" $Y^\mu(\sigma^\alpha, \theta)$. A superfield can be expanded as

$$Y^\mu(\sigma^\alpha, \theta) = X^\mu(\sigma^\alpha) + \bar{\theta}\psi^\mu(\sigma^\alpha) + \frac{1}{2}\bar{\theta}\theta F^\mu(\sigma^\alpha), \tag{3}$$

  where F is an auxiliary field which helps realizing the SUSY algebra off-shell. Question: why there are no linear-in-θ terms in the above expansion? [Hint: show that for a Majorana fermion ψ we have $\bar{\psi}\theta = \bar{\theta}\psi$.] See pages 113–118 of Becker-Becker-Schwarz for more on superspace.

3. Super-Virasoro, the constraint equations, and GSO
(suggested reading: Becker-Becker-Schwarz, GSW, and Johnson's *D-branes* book)

In the bosonic string theory, the equation of motion for the worldsheet metric implies conformal invariance, which allows choosing a light-cone gauge yielding a manifestly positive-norm spectrum. (Question: do you remember what happens with "spurious states"?) In the superstring theory, a supersymmetric conformal (or *superconformal*) invariance arises, which again allows a light-cone gauge analysis.

- Write down the super-Virasoro algebra arising in the NS sector of the superstring worldsheet. Check that its global part closes (this is called the $SU(1,1|1)$ algebra). Write down the corresponding "physical state conditions".
- Write down the super-Virasoro algebra arising in the R sector of the superstring worldsheet. Check that there is no closed global part in there that contains F_0 (the zero-mode of the supercurrent). Write down the corresponding "physical state conditions" (aka "constraint equations").
- Check that the Ramond ground state is degenerate, and the states in it form a spacetime fermion with 32 components. Construct the 32-dimensional representation by acting on a chosen ground state with 5 spinor creation operators formed from the Ramond fermion zero mode. (Do not use Gamma matrices here; they are not Hilbert space operators!) [The answer is given in Johnson's book, pp. 158,159.] Show that the F_0 equation (aka the Dirac-Ramond equation) removes half of the degrees of freedom, leaving 16 components. Another half is of course thrown away by the GSO projection, leaving 8 components (which are, depending on the GSO chirality, in $\mathbf{8}_c$ or $\mathbf{8}_s$ of the spacetime little group $SO(8)$).
- Find the tachyonic and massless spectrum of the closed superstring before the GSO projection.
- Show that the product of two vector representations of $SO(N)$ decomposes to a scalar, an anti-symmetric tensor, and a traceless symmetric tensor. Use the decomposition of $\mathbf{8}_v \times \mathbf{8}_v$ to deduce that the massless NS-NS sector of the closed superstring has a dilaton, a Kalb-Ramond field $B_{\mu\nu}$, and a metric $G_{\mu\nu}$.
- Argue that the decomposition of $\mathbf{8}_s \times \mathbf{8}_s$ and $\mathbf{8}_c \times \mathbf{8}_c$ are the same as that of $\mathbf{8}_v \times \mathbf{8}_v$. Show that the decomposition for the product of $\mathbf{8}_c \times \mathbf{8}_s$ reads $\mathbf{8} + \mathbf{56}$. [The required machinery is described in section 4.5 of 0506011.]
- Use the GSO projection along with the decompositions discussed above, to find the type IIA and IIB closed superstring massless spectra. These have a low-energy description as IIA and IIB supergravity theories.
- Show that the spectra of IIA and IIB supergravity theories become the same upon dimensional reduction on a spatial circle. [Hint: use Hodge duality for the bosonic parts; for the spinorial parts, argue that the differing chiralities in ten dimensions are not important after reduction to 9d where there is no chirality (see section 1.7

of Pope's Kaluza-Klein lecture http://people.physics.tamu.edu/ pope/ihplec.pdf for more on the reduction of fermions).]

4. **Supergravity, p-branes, and D-branes**
(suggested reading: Freedman-van Proeyen and Maldacena's thesis)

Let us focus on IIB supergravity. One of its equations of motion (omitting the derivative terms in the dilaton for simplicity)

$$R_{\mu\nu} = \frac{1}{4}H_{\mu\rho\sigma}H_{\nu}{}^{\rho\sigma} + e^{2\phi}\left(F_{1\mu}F_{1\nu} + \frac{1}{4}\tilde{F}_{3\mu\rho\sigma}\tilde{F}_{3\mu}{}^{\rho\sigma} + \frac{1}{24}\tilde{F}^{+}_{5\mu\rho\sigma\tau\kappa}\tilde{F}^{+\,\rho\sigma\tau\kappa}_{5\nu}\right). \qquad (4)$$

- Show that the 3-brane solution with $B = A_2 = C = \phi = 0$, and

$$\begin{aligned} \mathrm{d}s^2 &= H^{-1/2}(y)\mathrm{d}x^2 + H^{1/2}(y)\mathrm{d}y^2, \\ F_5 &= *_6\mathrm{d}H, \\ H &= 1 + \frac{L^4}{y^4}, \end{aligned} \qquad (5)$$

solves the IIB equation of motion; the coordinates x_i ($i = 0, 1, 2, 3$) are along the brane, and the y_j are perpendicular to it; the Hodge operator $*_6$ dualizes forms on the space of y_js. This is an extremal black brane solution with "Schwarzschild radius" L.

- Use the D3-brane tension equation to argue that if the above p-brane solution is sourced by N D3-branes, it would have $L^4 \propto N g_s \alpha'^2$. So the horizon radius is too small for the solution to represent a useful blackhole. (Note that increasing N doesn't help, because then the effective coupling near the horizon will be Ng_s, so the theory will be strongly coupled and hence not very useful still.)

- Use the gravitino supersymmetry variation in IIB supergravity to show that the SUSY generators ε^k_R ($k = 1, 2$) of the IIB theory should satisfy

$$\Gamma^0 \cdots \Gamma^3 \varepsilon^1_R = \varepsilon^2_R, \qquad (6)$$

for the 3-brane solution to be supersymmetric; so one spinor is determined in terms of the other. The 3-brane background thus breaks half of the supersymmetries of the theory (or is "half BPS"). [The result can also be found in Eq. (25.128) of the 2nd edition of Ortin's Gravity and Strings; note the typo in Eq. (25.127) there, in which "odd" and "even" should be swapped.]

The equations that are derived from the vanishing constraint on the SUSY variations of the supergravity (SUGRA) fermionic fields are often called *Killing spinor equations*. For the brane backgrounds of interest to us, no Killing spinor equation arises from the dilatino field; only the gravitino field yields non-trivial Killing spinor constraints. The equation for the p-brane background with general p (odd, for IIB, and between 1 and 9 of course) reads [again you can find it in Ortin's book; note that $\Gamma^{0\cdots p}$ used there is the same as $\Gamma^0\cdots\Gamma^p$ (why?); also the hats in there only mean that the objects are ten dimensional]

$$\Gamma^0\cdots\Gamma^p\varepsilon_R^1=\varepsilon_R^2. \tag{7}$$

- Show that a Dp-D$(p+4)$ system (such as a D1-D5 system) preserves 1/4 of the IIB supersymmetries (or is "quarter BPS"), but a Dp-D$(p+2)$ system (such as D1-D3) breaks all the supersymmetries. [Hint: write (7) for p and $p+2$; derive $\Gamma^0\cdots\Gamma^p\varepsilon_R^2=\Gamma^0\cdots\Gamma^{p+2}\varepsilon_R^2$; multiply both sides by $\Gamma^0\cdots\Gamma^p$ and use the fact that $(\Gamma^0\cdots\Gamma^p)^2=(-1)^{(p-1)/2}$ (can you see why?) to arrive at $\varepsilon_R^2=(-1)^{(p-1)/2}\Gamma^{p+1}\Gamma^{p+2}\varepsilon_R^2$. Finally use $(\Gamma^{p+1}\Gamma^{p+2})^2=-1$ to show that the Killing spinor equation can only be true if $\varepsilon_R^2=0$, meaning that SUSY is completely broken. Similarly, use $(\Gamma^{p+1}\Gamma^{p+4})^2=+1$ (why?) to show that the Killing spinor equation for the Dp-D$(p+4)$ system has non-trivial solutions, and more precisely that the D$(p+4)$ Killing spinor equation breaks the SUSY preserved by Dp down to one-quarter.]
- The p-brane solution corresponding to the D1-D5 system, placed on the background $R^2\times R^4\times T^4$, has the explicit metric, dilaton, and 3-form flux of IIB supergravity (we follow the notation in arXiv:hep-th/9702050)

$$\begin{aligned} e^{-2\phi}&=f_5/f_1,\\ \mathrm{d}s^2&=f_1^{-1/2}f_5^{-1/2}\mathrm{d}x_{||}^2+f_1^{1/2}f_5^{1/2}(\mathrm{d}r^2+r^2\mathrm{d}\Omega_3^2)+f_1^{1/2}f_5^{-1/2}\mathrm{d}x_{M_4}^2,\\ H_3&=2r_5^2\epsilon_3+2r_1^2e^{-2\phi}*_{10}\epsilon_7,\\ f_i&:=1+r_i^2/r^2\qquad i=1,5, \end{aligned}\tag{8}$$

  where $\mathrm{d}x_{||}^2=-\mathrm{d}t^2+\mathrm{d}x^2$ is the metric on the R^2, with x the coordinate along the D1-brane, and r parameterizing the radial coordinate on the R^4. The forms ϵ_3 and ϵ_7 are the volume forms of the three-cycle and the seven-cycle at $r=1$ inside the $R^4\times T^4$; the cycles can be alternatively described as the unique three-cycle

$\mathcal{C}_3$ and seven-cycle $\mathcal{C}_7$ inside $S^3 \times T^4$. Show that the IIB supergravity equation (4) is satisfied by this solution. D-brane tension analysis shows that $r_5^2 = g\alpha' Q_5$ and $r_1^2 = g\alpha' Q_1/v$, where $v = \mathrm{vol}(T^4)/(2\pi)^4\alpha'^2$, and Q_1, Q_5 are respectively the number of D1 and D5 branes. Again, the black p-brane Schwarzschild radius is small unless Q_1 and Q_5 are very large. But this time, unlike the 3-brane case, we get large blackholes with nice semi-classical description (see Maldacena's phd thesis for instance.)

- Show that the near horizon geometry of the 3-brane solution in (5) contains an AdS_5 factor. Show that the near horizon geometry of the p-brane solution of the D1-D5 system contains an AdS_3 factor. These are two of the most important tips of the AdS/CFT iceberg.