Free - particle Solutions of the Dirac Equation

$$\Psi(x) = u(p)\, e^{-ip\cdot x} \qquad p^2 = m^2$$

$$(\gamma^\mu p_\mu - m)\, u(p) = 0$$

$$p = p_0 = (m, 0) \xrightarrow{\Lambda_{\frac{1}{2}}} \text{general momentum}$$

$$(m\gamma^0 - m)\, u(p_0) = m \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} u(p_0) = 0$$

$$u(p_0) = \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix} \qquad \xi^\dagger \xi = 1$$

در حالت آزاد

مطابق اسپین

How does ξ behave under rotation?

$\xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \leftarrow$ spin up

بی نهایت کوچک در جهت z

$$\begin{bmatrix} E \\ p^3 \end{bmatrix} = \left(1 + \eta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) \begin{bmatrix} m \\ 0 \end{bmatrix}$$

$\eta \to$ محدود

$$\begin{pmatrix} E \\ p^3 \end{pmatrix} = \mathrm{Exp}\left[\eta \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right] \begin{pmatrix} m \\ 0 \end{pmatrix} = \left[\cosh\eta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sinh\eta \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right] \begin{pmatrix} m \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} m \sinh\eta \\ m \sinh\eta \end{pmatrix} \qquad \text{additive under successive boosts}$$

$\hookrightarrow$ rapidity

$$u(p) = \mathrm{Exp}\left[-\frac{\eta}{2} \begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix} \right] \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix}$$

به علامت و $\frac{1}{2}$ دقت کنیم

کلی

$$= \left[\cosh\frac{\eta}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \sinh\frac{\eta}{2} \begin{bmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{bmatrix} \right] \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix}$$

$$= \begin{pmatrix} \left[\sqrt{E+p^3}\left(\frac{1-\sigma^3}{2}\right) + \sqrt{E-p^3}\,\frac{1+\sigma^3}{2} \right]\xi \\ \\ \left[\sqrt{E+p^3}\left(\frac{1+\sigma^3}{2}\right) + \sqrt{E-p^3}\left(\frac{1-\sigma^3}{2}\right) \right]\xi \end{pmatrix}$$

$$u(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\;\xi \\ \\ \sqrt{p\cdot\bar{\sigma}}\;\xi \end{pmatrix}$$

کلی

p

یادآوری از اسرد

$$\psi_{p,\sigma} \triangleq N(p)\, U(L(p))\, \psi_{k,\sigma}$$

$$U(\Lambda)\, \psi_{p,\sigma} = N(p)\, U(L(\Lambda p)\, L^{-1}(\Lambda p)\, \Lambda L(p))\, \psi_{k,\sigma}$$

$$(p\cdot\sigma)(p\cdot\bar{\sigma}) = p^2 = m^2$$

$\xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

spin up ✓ $\quad u(p) = \begin{pmatrix} \sqrt{E-p^3}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \\ \sqrt{E+p^3}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} \xrightarrow{\text{large boost}} \sqrt{2E}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

$$\xi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$u(p) = \begin{pmatrix} \sqrt{E+p^3} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \\ \sqrt{E-p^3} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} \xrightarrow{\text{large boost}} \sqrt{2E} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

helicity

$$h \triangleq \hat{p} \cdot \vec{S} = \frac{\hat{p}_i}{2} \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}$$

$h = +\frac{1}{2}$ ← right-handed $\qquad$ $h = \frac{-1}{2}$ ← left-handed

آیا چپ دست برای هر مشاهده گر چپ دست است؟

Weyl Spinors

$$\underbrace{i\bar{\sigma} \cdot \partial \psi_L = 0} \qquad \underbrace{i\sigma \cdot \partial \psi_R = 0}$$

هلیسیتی مشخص

ناوردای لورنتس

$$u^\dagger u = \left(\xi^\dagger \sqrt{p\cdot\sigma}\, , \ \xi^\dagger \sqrt{p\cdot\bar{\sigma}}\right) \cdot \begin{pmatrix} \sqrt{p\cdot\sigma}\ \xi \\ \sqrt{p\cdot\bar{\sigma}}\ \xi \end{pmatrix} = 2E_p\, \xi^\dagger \xi$$

$$\bar{u}(p) = u^\dagger(p)\, \gamma^0$$

$$\bar{u}u = \left(\xi^\dagger \sqrt{p\cdot\sigma} \quad \xi^\dagger \sqrt{p\cdot\bar{\sigma}}\right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{p\cdot\sigma}\ \xi \\ \sqrt{p\cdot\bar{\sigma}}\ \xi \end{pmatrix} =$$

$$\xi^\dagger \Big[\underbrace{\sqrt{p\cdot\sigma}\sqrt{p\cdot\bar{\sigma}}}_{\sqrt{p^2} = m} + \underbrace{\sqrt{p\cdot\bar{\sigma}}\sqrt{p\cdot\sigma}}_{\sqrt{p^2} = m} \Big] \xi = 2m\, \xi^\dagger \xi$$

normalization $\xi^\dagger \xi = 1$

General solution of Dirac Eq. $\leftarrow$ linear combination of plane waves

$$\psi(x) = u(p)\, e^{-ip\cdot x} \qquad p^2 = m^2 \quad p^0 > 0$$

Two linearly independent solutions

$$u^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\ \xi^s \\ \sqrt{p\cdot\bar\sigma}\ \xi^s \end{pmatrix} \qquad \xi^s = 1, 2$$

$$\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\bar u^r(p)\, u^s(p) = 2m\delta^{rs} \qquad \equiv \qquad u^{r\dagger}(p)\, u^s(p) = 2E_p \delta^{rs}$$

فرکانس منفی

$$v^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\ \eta^s \\ -\sqrt{p\cdot\bar\sigma}\ \eta^s \end{pmatrix} \qquad s = 1, 2 \qquad \psi(x) = v(p)\, e^{ip\cdot x}$$

$p^2 = m^2 \quad p^0 > 0$

دقت

$$(i\gamma^\mu \partial_\mu - m)\psi = m \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} v(p_0) \qquad v(p_0) = \sqrt{m} \begin{pmatrix} \eta \\ -\eta \end{pmatrix}$$

$$\bar v^r(p)\, v^s(p) = -2m\delta^{rs} \qquad v^{r\dagger}(p)\, v^s(p) = +2E_p \delta^{rs}$$

$$\xi = \begin{pmatrix}0\\1\end{pmatrix} \begin{pmatrix}1\\0\end{pmatrix} \qquad \eta^s = \begin{pmatrix}0\\1\end{pmatrix} \begin{pmatrix}1\\0\end{pmatrix}$$

نشان دهید که (تمرین)

① $\bar{u}^r(p)\, v^s(p) = \bar{v}^r(p)\, u^s(p) = 0$

② $u^{r\dagger}(p)\, v^s(p) \neq 0 \qquad v^{r\dagger}(p)\, u^s(p) \neq 0$

③ $u^{r\dagger}(\vec{p})\, v^s(-\vec{p}) = v^{r\dagger}(-\vec{p})\, u^s(\vec{p}) = 0$

$$\sum_{s=1,2} u^s(p)\, \bar{u}^s(p) = \sum_s \begin{pmatrix}\sqrt{p\cdot\sigma}\, \xi^s \\ \sqrt{p\cdot\bar\sigma}\, \xi^s\end{pmatrix} \left(\xi^{s\dagger} \sqrt{p\cdot\bar\sigma}\, ,\ \xi^{s\dagger} \sqrt{p\cdot\sigma} \right)$$

$$= \cancel{\sqrt{p\cdot\sigma}} \qquad \sum_{1,2} \xi^s \xi^{s\dagger} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$$\sum_{s=1,2} u^s(p)\, \bar{u}^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\sqrt{p\cdot\bar\sigma} & \sqrt{p\cdot\sigma}\sqrt{p\cdot\sigma} \\ \sqrt{p\cdot\bar\sigma}\sqrt{p\cdot\bar\sigma} & \sqrt{p\cdot\bar\sigma}\sqrt{p\cdot\sigma} \end{pmatrix} = \gamma\cdot p + m$$

$$\sum_{1,2} \eta^s \eta^{s\dagger} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

Similarly

$$\sum_{s=1,2} v^s(p)\, \bar{v}^s(p) = \begin{bmatrix} -\sqrt{p\cdot\sigma}\sqrt{p\cdot\bar\sigma} & \sqrt{p\cdot\sigma}\sqrt{p\cdot\sigma} \\ \sqrt{p\cdot\bar\sigma}\sqrt{p\cdot\bar\sigma} & -\sqrt{p\cdot\bar\sigma}\sqrt{p\cdot\sigma} \end{bmatrix} = \gamma\cdot p - m$$

$\gamma\cdot p = \gamma^\mu p_\mu = \not{p} \leftarrow$ Feynman

$\bar{\psi}\psi$ ← جرم‌شاری تحت تبدیلات لورنتس ؟

$\bar{\psi}\gamma^{\mu}\psi$ ؟ ~ ~ ~ ~

$\bar{\psi}\Gamma\psi$ $\quad \Gamma \leftarrow 4\times 4$

پایه‌ی کامل

1	1
γ^{μ}	4
$\gamma^{\mu\nu} = \frac{1}{2}[\gamma^{\mu},\gamma^{\nu}] = \gamma^{[\mu}\gamma^{\nu]} = -i\sigma^{\mu\nu}$	6
$\gamma^{\mu\nu\rho} = \gamma^{[\mu}\gamma^{\nu}\gamma^{\rho]}$	4
$\gamma^{\mu\nu\rho\sigma} = \gamma^{[\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma]}$	1

$+ \quad 16$

تبدیلات تحت لورنتس

$$\bar{\psi}\gamma^{\mu\nu}\psi \longrightarrow (\bar{\psi}\Lambda^{-1}_{\frac{1}{2}})\left(\frac{1}{2}[\gamma^{\mu},\gamma^{\nu}]\right)(\Lambda_{\frac{1}{2}}\psi)$$

$$= \frac{1}{2}\bar{\psi}\left(\Lambda^{-1}_{\frac{1}{2}}\gamma^{\mu}\Lambda_{\frac{1}{2}}\Lambda^{-1}_{\frac{1}{2}}\gamma^{\nu}\Lambda_{\frac{1}{2}} - \Lambda^{-1}_{\frac{1}{2}}\gamma^{\nu}\Lambda_{\frac{1}{2}}\Lambda^{-1}_{\frac{1}{2}}\gamma^{\mu}\Lambda_{\frac{1}{2}}\right)\psi$$

$$= \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}\,\bar{\psi}\gamma^{\alpha\beta}\psi$$

$$\gamma^{5} \triangleq i\gamma^{0}\gamma^{1}\gamma^{2}\gamma^{3} = \frac{-i}{4!}\epsilon^{\mu\nu\sigma\rho}\gamma_{\mu}\gamma_{\nu}\gamma_{\rho}\gamma_{\sigma}$$

$$\boxed{\gamma^{\mu\nu\rho\sigma} = -i\epsilon^{\mu\nu\rho\sigma}\gamma^5 \qquad \gamma^{\mu\nu\rho} = i\epsilon^{\mu\nu\rho\sigma}\gamma_\sigma\gamma^5}$$

نشان دهید که

نشان دهید که

$$(\gamma^5)^\dagger = \gamma^5$$

$$(\gamma^5)^2 = 1$$

$$\{\gamma^5, \gamma^\mu\} = 0$$

تحت تبدیلات لورنتز γ^5 چگونه رفتار می‌کند؟

$$[\gamma^5, \cancel{\gamma} S^{\mu\nu}] = ?$$

نوتاسیون بسکین

$$\gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\gamma^5\psi_L = -\psi_L \qquad \gamma^5\psi_R = \psi_R$$

$$[\gamma^5, S^{\mu\nu}] = \ldots$$

Schur's lemma for reducibility

$$e^{-iS^{\mu\nu}\omega_{\mu\nu}}\gamma^5\psi_L = \gamma^5 e^{-iS^{\mu\nu}\omega_{\mu\nu}}\psi_L = -e^{-iS^{\mu\nu}\omega_{\mu\nu}}\psi_L$$

$$e^{-iS^{\mu\nu}\omega_{\mu\nu}}\gamma^5\psi_R = \gamma^5 e^{-iS^{\mu\nu}\omega_{\mu\nu}}\psi_R = e^{-iS^{\mu\nu}\omega_{\mu\nu}}\psi_R$$

$j^{\mu}(x) = \bar{\psi}\gamma^{\mu}\psi$ ↑ vector

$j^{\mu 5} = \bar{\psi}\gamma^{\mu}\gamma^{5}\psi$ ↑ pseudo-vector

$$\partial_{\mu} j^{\mu} = \partial_{\mu}\bar{\psi}\gamma^{\mu}\psi + \bar{\psi}\gamma^{\mu}\partial_{\mu}\psi = im\bar{\psi}\psi + \bar{\psi}(-im\psi) = 0$$

$$\partial_{\mu}\,\bar{\psi}_1\gamma^{\mu}\psi_2 = ?$$

پایه کامل

1	scalar	1
γ^{μ}	vector	4
$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^{\mu},\gamma^{\nu}]$	tensor	6
$\gamma^{\mu}\gamma^{5}$	pseudo-vector	4
γ^{5}	pseudo-scalar	1
		16

$$\partial_{\mu} j^{\mu 5} = 2im\bar{\psi}\gamma^{5}\psi$$

for $m=0$ axial vector current

Noether currents

$\psi(x) \to e^{i\alpha}\psi(x)$

$\psi(x) \to e^{i\alpha\gamma^{5}}\psi(x)$ ↑ تبدیلات کایرال

$\bar{\psi}\gamma\psi \to$

$\bar{\psi}\psi \to ?$

$\bar{\psi}\gamma^{5}\psi \to ?$

Fierz _ identities

$$(\sigma^{\mu})_{\alpha\beta}\,(\sigma_{\mu})_{\gamma\delta} = 2\,\varepsilon_{\alpha\gamma}\,\varepsilon_{\beta\delta}$$

به تبدیلات لورنتس آن توجه کنید.

$$(\bar{u}_{1R}\,\sigma^{\mu}\,u_{2R})(\bar{u}_{3R}\,\sigma_{\mu}\,u_{4R}) = 2\,\varepsilon_{\alpha\gamma}\,\bar{u}_{1R\alpha}\,\bar{u}_{3R\gamma}\,\varepsilon_{\beta\delta}\,u_{2R\beta}\,u_{4R\delta}$$

exchange

$$= -(\bar{u}_{1R}\,\sigma^{\mu}\,u_{4R})(\bar{u}_{3R}\,\sigma_{\mu}\,u_{2R})$$

نکته ی ظریف

$$(\bar{\sigma}^{\mu})_{\alpha\beta}\,(\bar{\sigma}_{\mu})_{\gamma\delta} = 2\,\varepsilon_{\alpha\gamma}\,\varepsilon_{\beta\delta}$$

$$(\bar{u}_{1L}\,\bar{\sigma}^{\mu}\,u_{2L})(\bar{u}_{3L}\,\bar{\sigma}_{\mu}\,u_{4L}) = -(\bar{u}_{1L}\,\bar{\sigma}^{\mu}\,u_{4L})(\bar{u}_{3L}\,\bar{\sigma}_{\mu}\,u_{2L})$$

چرا برای چپ ها $\bar{\sigma}_{\mu}$ بکار برده برای راست σ_{μ} ؟

یک اتحاد دیگه

$$\varepsilon_{\alpha\beta}\,(\sigma^{\mu})_{\beta\gamma} = (\bar{\sigma}^{\mu T})_{\alpha\beta}\,\varepsilon_{\beta\gamma}$$

$$\bar{\sigma}^{\mu}\sigma_{\mu} = 4$$

$$(\bar{u}_{1L}\,\bar{\sigma}^{\mu}\,\sigma^{\nu}\,\bar{\sigma}^{\lambda}\,u_{2L})(\bar{u}_{3L}\,\bar{\sigma}_{\mu}\,\sigma_{\nu}\,\bar{\sigma}_{\lambda}\,u_{4L}) = 16(\bar{u}_{1L}\,\bar{\sigma}^{\mu}\,u_{2L})(\bar{u}_{3L}\,\bar{\sigma}_{\mu}\,u_{4L})$$

چهارسولفای ← تمرین 3-6

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$$

$$H = \int d^3x \; \bar{\psi}(-i\vec{\gamma}\cdot\vec{\nabla} + m)\psi = \int d^3x \; \psi^\dagger[-i\gamma^0\vec{\gamma}\cdot\vec{\nabla} + m\gamma^0]\psi$$

$$\vec{\alpha} = \gamma^0\vec{\gamma} \qquad \beta = \gamma^0$$

Dirac Hamiltonian of one-particle quantum mechanics

$$h_D = -i\vec{\alpha}\cdot\vec{\nabla} + m\beta$$

Quantizing IS NOT so easy

Wrong way

$$[\psi_a(x), \psi_b^\dagger(y)] = \delta^{(3)}(x-y)\,\delta_{ab} \qquad \text{(زمان ساری)}$$

اندیس‌های اسپینوری

اگر ψ حقیقی بود مشکل پیش نمی‌آمد ولی نیست. اما خواهیم دید مشکلات اساسی‌تر داریم.

نمایشی این روابط جابجایی را برحسب a و $a^\dagger$ که H را قطری کند.

معادله‌ی حرکت

$$[i\gamma^0\partial_0 + i\vec{\gamma}\cdot\vec{\nabla} - m]\,u^s(p)\,e^{-ip\cdot x} = 0$$

$$\underbrace{i\gamma^0\partial_0 + i\vec{\gamma}\cdot\vec{\nabla}}_{i\gamma^\mu\partial_\mu}$$

$$\left.\begin{aligned} h_D\, u^s(p)\, e^{i\vec{p}\cdot\vec{x}} &= E_p\, u^s e^{i\vec{p}\cdot\vec{x}} \\ h_D\, v^s(p)\, e^{-i\vec{p}\cdot\vec{x}} &= -E_p\, v^s(p)\, e^{-i\vec{p}\cdot\vec{x}} \end{aligned}\right\} \rightarrow \psi \text{ Complete set}$$

$$v^s(\vec{p})\, e^{-i\vec{p}\cdot\vec{x}} \equiv v^s(-\vec{p})\, e^{i\vec{p}\cdot\vec{x}}$$

$$[Q, \mathcal{H}] = 0$$

$$[Q, \varphi] = -q\,\Phi$$

$$[Q, a] = -q\,a$$

$$[Q, a^\dagger] = q\,a^\dagger$$

$$[Q, a^{c\dagger}] = -q\,a$$

$$[Q, a^c] = q\,a^c$$

ص ۱۰
با اثر

ص ۱۹

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} e^{i\vec{p}\cdot\vec{x}} \sum_{s=1,2} \left(a_p^s u^s(p) + b_{-p}^{s\dagger} v^s(-p) \right)$$

Let us postulate

$$[a_p^r, a_q^{s\dagger}] = [b_q^r, b_q^{s\dagger}] = (2\pi)^3 \delta^3(p-q)\,\delta^{rs}$$

$$[\psi(x), \psi^\dagger(y)] = \int \frac{d^3p\,d^3q}{(2\pi)^6} \frac{e^{i(\vec{p}\cdot\vec{x} - \vec{q}\cdot\vec{y})}}{\sqrt{2E_p 2E_q}}$$

$$\sum_{r,s} \left([a_p^r, a_q^{s\dagger}]\, u^r(\vec{p})\, \bar{u}^s(q) + [b_{-p}^r, b_{-q}^{s\dagger}]\, v^r(-p)\, \bar{v}^s(-q) \right) \gamma^0$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{ip\cdot(x-y)} \left[(\gamma^0 E_p - \vec{\gamma}\cdot\vec{p} + m) + (\gamma^0 E_p + \vec{\gamma}\cdot\vec{p} - m) \gamma^0 \right]$$

$$= \delta^3(x-y)\, \mathbb{1}_{4\times 4}$$

$$H = \int \frac{d^3p}{(2\pi)^3} \sum_s \left(E_p\, a_p^{s\dagger} a_p^s - E_p\, b_p^{s\dagger} b_p^s \right)$$

اضافه کردن یا دزدن انرژی رو پایین !

$b \leftrightarrow b^\dagger$ رابطه‌ی جابجایی الزامی

$\int d^3x\, \psi^\dagger(x)\, \psi(x) \rightarrow$ ؟ چیست

ناجور شد !

حالا بحث علّیت

$[\psi(x), \psi^\dagger(y)]$ $\qquad [\psi(x), \bar\psi(y)]$

خارج

از مخروط نور

Heisenberg picture

$$e^{iHt} a_p^s e^{-iHt} = a_p^s e^{-iE_p t} \qquad H a_p = a_p (H - E_p)$$

$$e^{iHt} b_p^s e^{-iHt} = b_p^s e^{iE_p t} \qquad H b_p = b_p (H + E_p)$$

Heisenberg picture

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left(a_p^s u^s(p) e^{-ip\cdot x} + b_p^{s} v^s(p) e^{ip\cdot x} \right)$$

$$\bar\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left(a_p^{s\dagger} \bar u^s(p) e^{ip\cdot x} + b_p^{s\dagger} \bar v^s(p) e^{-ip\cdot x} \right)$$

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$$[\psi_a(x), \bar{\psi}_b(y)] = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \sum_s \Big(u_a^s(p)\, \bar{u}_b^s(p)\, e^{-ip\cdot(x-y)} + v_a^s(p)\, \bar{v}_b^s(p)\, e^{ip\cdot(x-y)} \Big)$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \Big((\not{p}+m)_{ab}\, e^{-ip\cdot(x-y)} + (\not{p}-m)_{ab}\, e^{ip\cdot(x-y)} \Big)$$

$$= (i\not{\partial}_x + m)_{ab} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \Big(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)} \Big)$$

$$= (i\not{\partial}_x + m)_{ab} [\phi(x), \phi(y)] \xrightarrow{\text{outside light cone}} 0$$

$$a_p^s |0\rangle = b_p^s |0\rangle$$

$$[\psi_a(x), \bar{\psi}_b(y)] = \langle 0 | [\psi_a(x), \bar{\psi}(y)] | 0\rangle$$

$$= \underbrace{\langle 0 | \psi_a(x) \bar{\psi}_b(y) | 0\rangle}_{\text{positive energy}} - \cancel{\langle 0 | \bar{\psi}_b(y) \psi_a(x) | 0\rangle}$$

negative ~

$$[a_p^r, a_q^{s\dagger}] = \cancel{[b_p^r, b_q^{s\dagger}]} = (2\pi)^3 \delta^3(p-q)\, \delta^{rs}$$

$$\cancel{b_p^s |0\rangle = 0}$$

اما حل را نذکرداریم

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \Big(a_p^s u^s(p) e^{-ip\cdot x} + b_p^s v^s(p) e^{ip\cdot x} \Big)$$

$$\bar{\psi}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \Big(a_p^{s\dagger} \bar{u}^s(p) e^{ip\cdot x} + b_p^{s\dagger} \bar{v}(p) e^{-ip\cdot x} \Big)$$

positive energy $\equiv$ negative frequencecy

$$\langle 0|\psi(x)\bar{\psi}(y)|0\rangle = \langle 0| \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_r a_p^r u^r(p) e^{-ip.x}\int\frac{d^3q}{(2\pi)^3}$$

positive energy

propagating from y to x

$$\frac{1}{\sqrt{2E_q}}\sum_s a_q^{st}\bar{u}^s(q)\, e^{iqy}|0\rangle$$

$$|0\rangle = e^{i\vec{P}.\vec{X}}|0\rangle$$

$$\langle 0|a_q^r a_q^{st}|0\rangle = \langle 0|a_p^r a_q^{st} e^{i\vec{P}.\vec{X}}|0\rangle = e^{i(\vec{p}-\vec{q}).\vec{x}}\langle 0|e^{i\vec{P}.\vec{X}} a_p^r a_q^{st}|0\rangle$$

$$= e^{i(\vec{p}-\vec{q}).\vec{x}}\langle 0|a_p^r a_q^{st}|0\rangle$$

$$\langle a_p^r a_q^{st}\rangle \neq 0 \Leftrightarrow \vec{p} = \vec{q}$$

$$e^{i\vec{J}.\theta}|0\rangle \qquad \langle a_p^r a_q^{st}\rangle \propto \delta^{rs}$$

$$\langle 0|a_p^r a_q^{st}|0\rangle = (2\pi)^3\delta^3(p-q)\,\delta^{rs} A(p) \qquad A(p) > 0$$

$$\langle 0|\psi(x)\bar{\psi}(y)|0\rangle = \int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}(\not{p}+m)\, A(p)\, e^{-ip.(x-y)}$$

invariant

$$A(p) = A(p^2) = A(m^2) = A$$

$$\langle 0|\psi_a(x)\bar{\psi}_b(y)|0\rangle = (i\not{\partial}_x+m)_{ab}\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}e^{-ip.(x-y)}\, A$$

$\langle 0|\bar{\psi}(y)\psi(x)|0\rangle$ $b|0\rangle \neq 0$

$a^\dagger|0\rangle = 0$

ی خطامی از positive energy سهم بگیرد .

همین خط استدلال

$$\langle 0|\bar{\psi}_b(y)\psi_a(x)|0\rangle = -(i\not{\partial}_x + m)_{ab}\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}e^{ip\cdot(x-y)} \quad B$$

$B>0$

$\langle 0|[\psi(x),\bar{\psi}(y)]|0\rangle \neq 0 \leftarrow$ outside light-cone

تعریف کنیم $A = B = 1$

$$\langle 0|\psi_a(x)\bar{\psi}_b(y)|0\rangle = -\langle 0|\bar{\psi}_b(y)\psi_a(x)|0\rangle$$

observables

energy charge particle number تعدادی ψ

$$[O_1(x), O_2(y)] = 0 \quad \text{for } (x-y)^2<0$$

$$\begin{cases} \{\psi_a(x), \psi_b^\dagger(y)\} = \delta^{(3)}(x-y)\delta_{ab} \\ \{\psi_a(x), \psi_b(y)\} = \{\psi_a^\dagger(x), \psi_b^\dagger(y)\} = 0 \end{cases}$$

$$\{a_p^r, a_q^{s\dagger}\} = \{b_p^r, b_q^{s\dagger}\} = (2\pi)^3\delta^3(p-q)\delta^{rs}$$

$$H = \int\frac{d^3p}{(2\pi)^3}\sum_s\left(E_p a_p^{s\dagger}a_p^s - E_p b_p^{s\dagger}b_p^s\right)$$

$$\{ b_p^r , b_q^{s\dagger} \} = (2\pi)^3 \delta^3(p-q) \delta^{rs}$$ ← b_q^r و $b_q^{s\dagger}$ متقارن

بازتعریف

$$\tilde{b}_p^s \doteq b_{p}^{s\dagger} \qquad \tilde{b}_p^{s\dagger} \doteq b_q^s$$

$$-E_p b_p^{s\dagger} b_p^s = E_p \tilde{b}_p^{s\dagger} \tilde{b}_p^s - \text{cte}$$

$$a_p^s |0\rangle = \tilde{b}_p^s |0\rangle = 0$$ ← انرژی مثبت

$$\{b, b^\dagger\} = 1 \qquad \{b, b\} = \{b^\dagger, b^\dagger\} = 0$$

$$|0\rangle \qquad b|0\rangle = 0 \qquad b^\dagger|0\rangle = |1\rangle \qquad b|1\rangle = |0\rangle \qquad b^\dagger|1\rangle = (b^\dagger)^2|0\rangle = 0$$

$|0\rangle$
$|1\rangle$
کدام را خلاء بنامیم؟

$$a_p^\dagger a_q^\dagger |0\rangle = -a_q^\dagger a_p^\dagger |0\rangle$$ ← آمار فرمی دیراک

ناوردایی لورنتس
انرژی‌های مثبت
نرم مثبت
علیت

پائولی ← فرمی دیراک ← این نحوه صحیح
نه صحیح ← بوز-اینشتین

$\bar{b} \to b$

میدان دیراک کوانتیده

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left(a_p^s u^s(p) e^{-ip\cdot x} + b_p^{s\dagger} v^s(p) e^{ip\cdot x} \right)$$

$$\bar{\psi}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left(b_p^s \bar{v}^s(p) e^{-ip\cdot x} + a_p^{s\dagger} \bar{u}^s(p) e^{ip\cdot x} \right)$$

$$\{a_p^r , a_q^{s\dagger}\} = \{b_p^r , b_q^{s\dagger}\} = (2\pi)^3 \delta^3(p-q)\, \delta^{rs}$$

$$\{a_p^r , b_q^r\} = 0 = \{a_p^r , b_q^{s\dagger}\} = \{a_p^r , a_q^s\} = \{b_p^r , b_q^s\}$$

$$\{\psi_a(x) , \psi_b^\dagger(y)\} = \delta^3(x-y)\, \delta_{ab}$$

$$\{\psi_a(x) , \psi_b(y)\} = \{\psi_a^\dagger(x) , \psi_b^\dagger(y)\} = 0$$

$$a_p^s |0\rangle = b_p^s |0\rangle = 0$$

$$H = \int \frac{d^3p}{(2\pi)^3} \sum_s E_p \left(a_p^{s\dagger} a_p^s + b_p^{s\dagger} b_p^s \right) + \infty$$

$$P = \int d^3x\, \psi^\dagger (-i\vec{\nabla})\psi = \int \frac{d^3p}{(2\pi)^3} \sum_s \vec{p} \left(a_p^{s\dagger} a_p^s + b_p^{s\dagger} b_p^s \right)$$

هم $a_p^{s\dagger}$ و هم $b_p^{s\dagger}$ ذرات با انرژی E_p و مومنتم $\vec{p}$ به وجود می آورند.

$a_p^{s\dagger}|0\rangle \to$ fermion

$b_p^{s\dagger}|0\rangle \to$ antifermion

$$|p,s\rangle \triangleq \sqrt{2E_p}\, a_p^{s\dagger}|0\rangle$$

$$\langle p,r|q,s\rangle = 2E_p (2\pi)^3 \delta^3(p-q)\,\delta^{rs}$$

ناوردای لورنتس

$U(\Lambda) \to$ unitary $\quad \Lambda_{\frac{1}{2}}$ is not unitary

$$U\psi(x)U^{-1} = U\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_s \left(a_p^s u^s e^{-ipx} + b_p^{s\dagger} v^s(p) e^{ip\cdot x}\right)U^{-1}$$

$\vec{S} \parallel$ محور دوران

$$U(\Lambda)\, a_p^s\, U^{-1}(\Lambda) = \sqrt{\frac{E_{\Lambda p}}{E_p}}\, a^s_{\Lambda p}$$

$$\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\, a_p^s = \underbrace{\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}}_{\text{Lorentz invariant}} \underbrace{\sqrt{2E_p}\, a_p^s}_{U} \; U^{-1}$$

$$U(\Lambda)\psi(x)U^{-1}(\Lambda) = \int \frac{d^3\tilde{p}}{(2\pi)^3}\frac{1}{2E_{\tilde{p}}}\sum_s u^s(\Lambda^{-1}\tilde{p})\sqrt{2E_{\tilde{p}}}\, a^s_{\tilde{p}}\, e^{-i\tilde{p}\cdot\Lambda x} + \cdots$$

$$\tilde{p} = \Lambda p$$

$$u^s(\Lambda^{-1}\tilde{p}) = \Lambda^{-1}_{\frac{1}{2}}\, u^s(\tilde{p})$$

$$U(\Lambda)\psi(x)U^{-1}(\Lambda) = \int \frac{d^3\tilde{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\tilde{p}}}}\sum_s \Lambda^{-1}_{\frac{1}{2}} u^s(\tilde{p})\, a^s_{\tilde{p}}\, e^{-i\tilde{p}\cdot\Lambda x} + \cdots$$

$$= \Lambda^{-1}_{\frac{1}{2}}\, \psi(\Lambda x)$$

تفاوت

$$\varphi(x) \to M_{ab}(\Lambda)\,\varphi_b(\Lambda^{-1}x)$$