Short Description of Topics in CFT course at IPM

Lecturer: ***M.M. Sheikh-Jabbari***

Conformal group is an extension of Poincaré group by dilatation and special conformal transformations. Conformal Field Theories (CFTs) are a special class of quantum field theories the action of which exhibit invariance under the conformal group. If the conformal symmetry is not anomalous, i.e. it is an exact symmetry of the quantum theory, the symmetry may be used to restrict the form of n-point functions of the theory.

CFTs can appear in various spacetime dimensions, e.g. one, two, three, four and six dimensions and the conformal algebra in dimension $d \geq 3$ is $so(d-1,2)$ while in two dimensions it is the infinite dimensional Virasoro algebra. Therefore, 2d CFTs are special and have been studied much more than the higher dimensional CFTs. CFTs in 4d have also been constructed and analyzed extensively; all known nontrivial (interacting) 4d CFTs are of the form of supersymmetric gauge theories. 3d CFTs have also been discussed and analyzed primarily through the AdS/CFT duality, however, in the recent five-six years an explicit action for the 3d CFTs, which are specific 3d supersymmetric gauge theories, has been constructed. The 6d CFTs are perhaps the less understood ones and we do not have an explicit action for them and our description of them is limited to either their Discrete Light-Cone Quantization (DLCQ) description, their compactification to lower dimensions and their gravity duals via AdS/CFT.

Another handle on CFTs, in general, comes from the fact that any QFT at its RG fixed point exhibits scaling symmetry. As such, any QFT may flow to a non-trivial (interacting) CFT, if such a non-trivial fixed point exists. In this viewpoint, 3d CFTs are known to arise from non-trivial IR fixed point of 3d Yang-Mills theories, while 6d CFT(s) is (are) arising from 5d Yang-Mills in the UV fixed point.

In this course we study and discuss the following topics in CFTs:

- Introduction of conformal algebra in various dimensions and their representations.

  5 sessions.

- General Remarks on CFT's.

  3 sessions.

- A brief introduction to 2d CFTs and their classification by central charge. We also discuss modular invariance of 2d CFTs and derive Cardy formula.

  5 sessions.

- A brief discussion on super-conformal algebras.

  1 sessions.

- On 4d (super)conformal theories.

2 sessions.

- On 3d and 6d CFTs.

1 sessions.

Important Notes for the Students

- This is an advanced PhD level course and I assume background knowledge of QFT at the level of Peskin-Schroder as well as basic group theory and their representation theory.

- During the lectures I will provide several questions and problems and registered students are supposed to solve and return them to me. The final grade will be based on them.

- **Texts and reading:**
  - *P. Di Francesco, P. Mathieu and D. Senechal, "Conformal Field Theory," New York, USA: Springer (1997).*
  - *S. Minwalla, "Restrictions imposed by Superconformal Invariance On Quantum Field Theories,"* `http://arxiv.org/pdf/hep-th/9712074.pdf`.
  - *V. Rychkov, "Lectures on Conformal Field Theory in Higher Dimensions ($D \geq 3$)),"* `https://sites.google.com/site/slavarychkov/`.
  - *Yu Nakayama, "A lecture note on scale invariance vs conformal invariance,"*, `http://arxiv.org/pdf/1302.0884v1.pdf`

- The above references contain many further references and reading material.

- This course is equivalent to *two units* and there is the possibility of formally registering for the course as a "guest student" for non-IPM students. For the latter please arrange the formal details with department office, Ms Pileroudi, niloufar@theory.ipm.ac.ir.

- The lectures will be **9-11, Saturday and Monday**, *in Farmanieh Bldg* and they will **start on 14th of Bahman (Feb. 2nd 2013)**.

Contents

1 General Remarks

- Standard (Q)FT assumes a background spacetime.
- This spacetime is usually assumed to be Minkowski (for relativistic FT's).
- We may have non-relativistic QFT's (e.g. in Cond.Mat.), e.g. with Galilean symmetry.
- Rel.QFT's are expected to exhibit Lorentz and spacetime translation invariance. The latter two form the Poincaré group.
- Poincaré group in $d+1$ dimensional spacetime is $ISO(d,1)$, which is the *isometry group* of the $d+1$ dim. Minkowski space.
- That is, Poincaré group can be viewed as a ***linear*** coordinate transformation which keeps the form of the Minkowski metric $\eta_{\mu\nu} = diag(-1,+1,+1,\cdots,+1)$ invariant.
- All physical observables of Rel.QFT's are hence required to be based on *Lorentz invariant* quantities, or fall into specific Lorentz representations.

  NOTE: *There are quantities like cross-sections and half life-times which are not Lorentz invariant, however, they are based on Lorentz-inv. S-matrix elements.*
- Since $dim(ISO(d,1)) = d(d+1)/2+d+1 = (d+1)(d+2)/2$, these isometries completely fix the metric to be $\eta_{\mu\nu}$.
- One may ask if it is possible to extend the (Minkowski) spacetime symmetries beyond Poincaré &
- if yes, whether it is possible to find QFT's which exhibit those symmetries.
- Being max. sym. group, one cannot naively extend the set of linear isometries of Poincaré group.
- Consider *infinitesimal* linear transformations

$$x_\mu \to x_\mu + \delta x_\mu\,, \qquad \delta x_\mu = A_{\mu\nu} x_\nu + b_\mu\,,$$

  - If $A_{\mu\nu}$ is antisymmetric we recover Lorentz transformations which keep $\eta_{\mu\nu}$ invariant.
  - Symmetric part of $A_{\mu\nu}$ may be brought to a diagonal form using a general Lorentz transformation. And hence, we only remain with

$$A_{\mu\nu} = diag(A_1, A_2, \cdots, A_d).$$

  These transformations will change metric $\eta_{\mu\nu}$.

– However, among them there is the special case, **scaling transformations** $A_{\mu\nu} = \lambda\eta_{\mu\nu}$ which takes

$$\eta_{\mu\nu} \to \lambda^{-2}\eta_{\mu\nu} \, .$$

– The scaling, although changes metric, it takes it to something proportional to it.

2 The conformal group

- So, we seem to have an idea on how to extend the notion of isometry:

  Coordinate Transformations (not necessarily linear) under which

$$\eta_{\mu\nu} \to f(x)\eta_{\mu\nu} \, . \tag{2.1}$$

- These transformations are called **conformal** transformations.

  ▶▶ **Exercise 2.1:** *Show these transformations form a group, the conformal group.*

- It is readily seen that the Poincaré is a subgroup of conformal group.

- One may consider the cases where the background metric is not conformally flat. This is an immediate generalization which is used in drawing Penrose diagrams.

- Scaling is also an element of the conformal group.

- **Active and passive conformal transformations:** In the above we defined "passive" conformal transformations, i.e. coordinate transformations such that:

$$x \to x' = x'(x) \, , \qquad g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = f(x)g_{\mu\nu}(x) \, , \tag{2.2}$$

  The above being a diffeomorphism does not change the physical length $ds^2 \to ds^2$.

  However, intuitively, we are more used to thinking about coordinates and the length, instead of coordinates and metric tensor. For examples, we usually think that scaling changes the physical distances and so on. In a more rigorous mathematical language, let us consider a general **Weyl scaling** (and not a diffeomorphism):

$$x \to x \, , \qquad g_{\mu\nu}(x) \to g'_{\mu\nu}(x) = f(x)g_{\mu\nu}(x) \implies ds^2 \to f(x)ds^2 \, . \tag{2.3}$$

  Weyl transformations obviously form a group.
  ▶▶ **Exercise 2.2:** *Show this.*

  Of course not all Weyl scalings produce a transformation like in (2.1), there is a specific set of Weyl scalings which has this property, this is the conformal group. In other words, *Conformal group consists of* **Diff** $\times$ **Weyl** *transformations subject to:*

$$x^\mu \to x'^\mu = x'^\mu(x)\,, \qquad g_{\mu\nu} \to g'_{\mu\nu}(x') = g_{\mu\nu}(x)\,. \tag{2.4}$$

▶▶ **Exercise 2.3:** *Show that with the above the physical length changes as in* (2.3).

▶▶ **Exercise 2.4:** *Convince yourself that the set of diffeomorphism s which are allowed in* (2.4) *are ONLY those appearing in* (2.1).

- Let's investigate (2.1) or equivalently (2.4) more closely. Consider infinitesimal coordinate transformations, *diffeomorphisms*, which generate (2.1):

$$x^\mu \to x^\mu + \xi^\mu(x)\,. \tag{2.5}$$

Question: Which $\epsilon_\mu(x)$ generates (2.4)?

- Recalling transformation of metric under a general diffeomorphism we learn that

$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = (f(x) - 1)\eta_{\mu\nu} \tag{2.6}$$

NOTE: *For a generic metric* $\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = h(x) g_{\mu\nu}$ *is called* ***conformal Killing vector*** *equation.*

- Trace of Eq.(2.6) gives

$$f(x) - 1 = \frac{2}{d}\partial_\mu \xi^\mu\,.$$

- Plugging the above into (2.6) and taking once more derivative leads to

$$(d-1)\partial^2 f(x) = 0\,, \qquad \partial^2 \xi_\mu = \frac{2-d}{2}\partial_\mu f\,. \tag{2.7}$$

▶▶ **Exercise 2.5:** *Work out the above.*

- One may then solve the above. Since $d = 2$ and $d > 2$ cases are quite different we consider the two cases separately.

- Let us consider the $d > 2$ case first. $d = 2$ will be discussed later.

$$\begin{gathered} \xi^\mu = A^\mu + B^\mu{}_\nu x^\nu + C^\mu{}_{\alpha\beta} x^\alpha x^\beta\,. \\ B_{\{\mu\nu\}} = A\eta_{\mu\nu}\,, \quad C_{\{\mu\nu\}\alpha} = B_\alpha \eta_{\mu\nu}\,, \quad C_{\mu\alpha\beta} = C_{\mu\beta\alpha}. \end{gathered} \tag{2.8}$$

▶▶ **Exercise 2.6:** *Work out the above .*

That is *infinitesimal* dilation and special conformal transformations are

$$x^\mu \to (1+\lambda)x^\mu\,, \qquad x^\mu \to x^\mu + 2(b\cdot x)x^\mu - x^2 b^\mu\,,$$

- for $d > 2$ one may solve (2.8) and see that:
  - A_μ generates rigid translations,
  - $B_{\mu\nu}$ generates Lorentz transformations and rigid scaling,
  - $C_{\mu\nu\alpha}$ generates "special conformal transformations".

  ▶▶ **Exercise 2.7:** *Show this.*

  ▶▶ **Exercise 2.8:** *Work out FINITE scaling and special conformal transformations. That is, find $x_\mu \to x'_\mu = x'_\mu(x_\nu, \lambda, b_\mu)$ where λ is parameterizing rigid scaling and b_μ the special conformal transformations.*

  ▶▶ **Exercise 2.9:** *If $x' = x'(x)$ then show that $f(x)^{-d} = \left|\frac{\partial x'}{\partial x}\right|$.*

  ▶▶ **Exercise 2.10:** *Show that*
  $$\left|\frac{\partial x'}{\partial x}\right| = \frac{1}{(1 - 2b\cdot x + b^2x^2)^d} \qquad \textit{for special conformal transformations}$$
  and
  $$\left|\tfrac{\partial x'}{\partial x}\right| = \lambda^{-d} \qquad \textit{for rigid scaling}.$$

  ▶▶ **Exercise 2.11:** *How does the "distance" between two points x_μ, y_μ, $|x-y|$, transform under conformal transformations.*

  ▶▶ **Exercise 2.12:** *Show that given three points x_1, x_2, x_3 one cannot construct conformally invariant combinations. While for four and more points* ***cross-ratios***
  $$\frac{|x_i - x_j||x_k - x_l|}{|x_i - x_k||x_j - x_l|}\,, i \neq j \neq k \neq l\,,$$
  are invariant under generic conformal transformations.
  Show that for N points there are $N(N-3)/2$ ***independent*** *cross-ratios.*

  ▶▶ **Exercise 2.13:** *Show that the group generated by the above is $SO(d,2)$.*
  HINT: To show the above, one may start with the conformal *algebra*, which is the (Lie) algebra generated by the Lie bracket of conformal diffeomorphisms:
  $$\xi = A^\mu \partial_\mu + B^{\mu\nu} x_\mu \partial_\nu + C^{\mu\nu\alpha} x_\mu x_\nu \partial_\alpha\,,$$
  i.e. computing $[\xi_1, \xi_2]$, where ξ_i are two conformal Killing vector fields generated by different A, B, C parameters.

  NOTE: *The conformal group in $d > 2$ is the largest FINITE dimensional subgroup of diffeomorphism s.*

- If we denote the generators by L_{IJ}, $I, J = -1, 0, 1, \cdots d$, then

- $L_{\mu\nu}$, $\mu, \nu = 0, 1, \cdots d-1$ are d dimensional Lorentz generators,
- $P_\mu = L_{-1\mu} + L_{d\mu}$ are d momentum (spacetime translations) generators.
  NOTE: *Hamiltonian is also among these.*
- $K_\mu = -L_{-1\mu} + L_{d\mu}$ are special conformal generators.
- $D = L_{-1,d}$ is the generator of rigid scaling, also called *dilation*.

▶▶ **Exercise 2.14:** *Starting from the $SO(d, 2)$ algebra*

$$[L_{AB}, L_{CD}] = i\left(g_{AC}L_{BD} + g_{BD}L_{AC} - g_{BC}L_{AD} - g_{AD}L_{BC}\right) , \tag{2.9}$$

where $g_{AB} = diag(-1, -1, +1, \cdots , 1)$, work out the commutation relations of $L_{\mu\nu}, P_\mu, K_\mu, D$.

▶▶ **Exercise 2.15:** *If we denote the d dimensional spacetime coordinates by x_μ, we now that $P_\mu = -i\partial_\mu$ and $L_{\mu\nu} = ix_\mu\partial_\nu - ix_\nu\partial_\mu$. Work out representation of D and K_μ in this basis.*

NOTE: *For Euclidean d dimensional space the conformal group is $SO(d + 1, 1)$.*

2.1 Representations of the conformal group, in $d > 2$

- Having an indefinite metric, the irreps of conformal group, like the Lorentz or Poincaré are infinite dimensional. That is we are dealing with "field theories".
- Noting that $ISO(d-1, 1) \in SO(d, 2)$, representations of conformal group may hence be constructed based upon the irreps of Poincaré group in the same spacetime dimension.
- Noting that $SO(d - 1, 1) \times SO(1, 1)_D$ is a maximal subgroup of the conformal group. All reps of the conformal group may be labeled by their Lorentz rep and their scaling dimension (eigenvalues of dilatation operator).
- Note that generators of momentum P_μ as well as P^2, do not commute with dilatation operator D,
- while generators of Lorentz and in particular rotations do commute with D. So,
- States of non-zero mass cannot have definite scaling dimension (eigenvalues of D).
- Reps of conformal group with non-zero mass should hence involve an *infinite* number of states with a *continuum* mass spectrum. This is not accounted for in a (standard) particle based QFT. This has been explored in the last five years under the title of "unparticle physics" [H. Georgi, Phys.Rev.Lett. 98 (2007) 221601.]
- In general, hence, states in the irreps of conformal group are labeled by their spin and scaling dimension and have zero mass (eigenvalue of P^2).

- As discussed irreps of conformal group consists of massless irreps of Poincaré algebra which are the standard fields (functions):

$$\Phi(x) \to \Phi'(x') = (1 + i\omega(x))\Phi(x)\,, \qquad x \to x'\ .$$

where $\omega(x)$ is a generic element of the conformal algebra.

- To find what are solutions to the above equation, i.e. the reps of conformal group, let's recall the construction of Poincaré group reps:
  - Among Poincaré transformations there are those which transform the origin $x_\mu = 0$ (like translations P_μ), and those which do not transform the origin, the Lorentz generators.
  - The Lorentz generators $J_{\mu\nu}$ have an *external* part $L_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu)$, which is ZERO at the origin, and an *internal* part $S_{\mu\nu}$.
  - The internal part generators commute with ALL external Poincaré generators:

$$[S_{\mu\nu}, P_\mu] = 0\,, \qquad [S_{\alpha\beta}, L_{\mu\nu}] = 0\,, \tag{2.10}$$

    while

$$[S_{\mu\nu}, S_{\alpha\beta}] = i(S_{\mu\alpha}\eta_{\nu\beta} + S_{\nu\beta}\eta_{\mu\alpha} - S_{\nu\alpha}\eta_{\mu\beta} - S_{\mu\beta}\eta_{\nu\alpha})\,. \tag{2.11}$$

- One may then focus on the "internal" part first, whose irreps are labeled by *spin*.
- Explicitly, for Poincaré irreps:

$$P_\mu\Phi(x) = -i\partial_\mu\Phi(x)\,, \qquad J_{\mu\nu}\Phi(x) = L_{\mu\nu}\Phi(x) + S_{\mu\nu}\Phi(x)\,.$$

- reps of the external parts are *infinite dimensional* while the internal parts are *finite dimensional*.

▶▶ **Exercise 2.16:** *Can we also associate an internal part of momentum P_μ?!*

- For the conformal group, adding D and K_μ we may proceed in the same way:
  - D has an external part which is $-ix^\mu\partial_\mu$ and an *internal* part denoted by $\boldsymbol{\Delta}$.
  - K_μ has an external part which is $-i(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu)$ and an *internal* part denoted by κ_μ.
- As in the Poincaré part, the external parts vanish at the origin.
- The internal parts $\boldsymbol{\Delta}$ and κ_μ, just like spin, are (finite dimensional) matrices.

▶▶ **Exercise 2.17:** *What is the subgroup of the the conformal group which keeps the origin invariant?*

- The *internal conformal algebra* is hence

$$[\boldsymbol{\Delta}, S_{\mu\nu}] = 0\,, \qquad [\boldsymbol{\Delta}, \kappa_\mu] = -i\kappa_\mu\,, \tag{2.12}$$
$$[\kappa_\mu, \kappa_\nu] = 0\,, \qquad [S_{\mu\nu}, \kappa_\alpha] = -i(\eta_{\mu\alpha}\kappa_\nu - \eta_{\nu\alpha}\kappa_\mu) \tag{2.13}$$

- Returning to reps:

$$D\Phi(x) = (-ix^\mu\partial_\mu + \boldsymbol{\Delta})\Phi(x)\,, \tag{2.14}$$
$$K_\mu\Phi(x) = \left[-i(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu) + 2x_\mu\boldsymbol{\Delta} - S_{\mu\nu}x^\nu + \kappa_\mu\right]\Phi(x)\,. \tag{2.15}$$

▶▶ **Exercise 2.18:** *Show the above.*

- IF we demand $\Phi(x)$ to be in an irrep of Lorentz group (i.e. to have a definite spin), then (2.12) implies it can have a definite $\boldsymbol{\Delta}$ too. ($\boldsymbol{\Delta}$ is identity matrix for this irrep).

NOTE: *Remember that $\boldsymbol{\Delta}$ is generator of $SO(1,1)$ and is not Hermitian. However, $-i\boldsymbol{\Delta}$ is Hermitian.*

NOTE: *Therefore, reps of dilation are not unitary.*

NOTE: *The eigenvalue of $-i\boldsymbol{\Delta}$ is called the* ***scaling dimension****.*

- (2.13) implies that $\kappa_\mu = 0$.

- For a *scalar field* $\phi(x)$ with scaling dimension Δ, under conformal transformations (2.4)

$$x_\mu \to x'_\mu\,, \qquad \eta_{\mu\nu} \to \eta_{\mu\nu}\,,$$
$$\phi(x) \to \phi'(x') = f(x)^\Delta\phi(x)\,. \tag{2.16}$$

That is, if we consider only the rigid scalings $x \to \lambda x$ (corresponding to $f(x) = \lambda^{-1}$), then

$$\phi'(\lambda x) = \lambda^{-\Delta}\phi(x)\,. \tag{2.17}$$

- Note that conformal transformations consists of an overlap of diffeomorphism and Weyl scaling. Therefore, as e.g. seen in (2.16), $\phi(x)$ behaves as an scalar under the diffeomorphism and the $f(x)^\Delta$ indicates the representation under Weyl scaling.

▶▶ **Exercise 2.19:** *Show* (2.17) *and work out $\delta\phi$ under scaling.*

- A field with given Δ like above, is called ***quasi-primary***.

NOTE: *In the existing literature and for $d > 2$ cases, may also be called* ***primary****. In 2d case, as we will see, there is a distinction between primary and quasi-primary.*

- A quasi-primary field/operator $\mathcal{O}(x)$ is hence a field for which $K_\mu \mathcal{O}|_{x=0} = 0$. That is, quasi-primary fields are killed by the special conformal generators K_μ.
- An operator which is **not killed** by K_μ while it has a **definite scaling dimension** is called a *descendent* operator.

►► **Exercise 2.20:** *Repeat the above for vector and tensor fields.*
HINT: For a quasi-primary field of generic spin (2.16) should replaced with

$$\Phi(x) \to \Phi'(x') = f(x)^{\Delta}\ \mathcal{R}[\Phi]\,. \tag{2.18}$$

where $\mathcal{R}$ is the representation of Lorentz group associated with Φ. For example for the case of vector field: $V'_\mu(x') = f(x)^{\Delta}\ \mathcal{R}_\mu^{\ \nu} V_\nu(x)$.

NOTE: *Here we are discussing the conformal group and algebra and its representations in various dimensions and how they are labeled. We will turn to the issue of conformal group as the symmetry group of a given QFT afterward.*

NOTE: *As discussed the conformal group is a very specific subgroup of $Diff \times Weyl$* (2.4)*. The prefactor $f(x)^{\Delta}$ in the above is basically indicating the representation of the Weyl scaling and being scalar, spinor, vector and ... determine the representation under diffeomorphism .*

►► **Exercise 2.21:** *Let $\mathcal{O}(x)$ be a local operator of dimension Δ and spin s, what is the dimension and spin of operators $P_\mu \mathcal{O}(0)$ and $K_\mu \mathcal{O}(0)$. What about $P_\mu P_\nu \mathcal{O}(0)$ and $K_\mu K_\nu \mathcal{O}(0)$. Answer the same question when $\mathcal{O}$ is a quasi-primary operator.*

►► **Exercise 2.22:** *Compute the second rank Casimir of the conformal group $\mathcal{L}^2 \equiv L_{AB}L^{AB}$. Compute $\mathcal{L}^2|_{x=0}\mathcal{O}(0)$ where $\mathcal{O}$ is a quasi-primary field of spin s and dimension Δ.*

NOTE: *We will return to the question of unitary representation of the conformal group in the next section after discussing the notion of radial quantization.*

2.2 Conformal algebra in 2d

- Conformal Killing equation for $d = 2$ is

$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = \partial_\alpha \xi^\alpha \eta_{\mu\nu}\,, \quad \partial^2 \xi_\mu = 0 \quad \Rightarrow \quad \xi^+ = \xi(t+x) \quad \textit{or} \quad \xi^- = \xi(t-x)\,. \tag{2.19}$$

For Euclidean version we have a similar expression:

$$z = x + iy\,, \quad \bar{z} = x - iy\,, \qquad \xi = \xi(z)\,, \quad \bar{\xi} = \bar{\xi}(\bar{z})\,. \tag{2.20}$$

▶▶ **Exercise 2.23:** *Work out the above .*

NOTE: *in the Euclidean case* $ds^2 = 2dzd\bar{z}$ *and under* $z \to f(z)$ *(ALL meromorphic transformations)* $ds^2 = 2|\partial f|^2 dzd\bar{z}$.

NOTE: *In our 2d analysis we usually treat* z *and* $\bar{z}$ *as two INDEPENDENT complex coordinates and at the end of computations take* $\bar{z}$ *to be complex conjugate of* z, z^*.

NOTE: *Having boundary conditions on the fields/functions will generically relate holomorphic and anti-holomorphic parts.*

- Therefore, $d = 2$ it is *infinite dimensional*, while in $d > 2$ the conformal group is *finite dimensional.*
- In 2d the conformal group is infinite dimensional as its generators are *all meromorphic diffeomorphism* $w = w(z)$.

  NOTE: *Here we discuss the Euclidean case, but the Minkowski case goes through almost in the same way. Using complex coordinates* $z, \bar{z}$ *is naturally associated with the light-cone coordinates in the Minkowski case.*
- Any such diffeomorphism admits a Laurent expansion

$$ds^2 = 2dwd\bar{w} = 2|f(z)|^2 dzd\bar{z}\,, \quad f(z) = \partial_z w\,, \quad w = \sum_{n\in Z} \xi_n z^n\,. \tag{2.21}$$

- Associated with each z^n there is a generator,

$$\xi_n = -z^{n+1}\partial_z, \qquad \bar{\xi}_n = -\bar{z}^{n+1}\partial_{\bar{z}}\,. \tag{2.22}$$

- The conformal algebra, as before, is nothing but the Lie bracket of the diffeomorphisms generating it:

$$[\xi_n, \xi_m] = (n-m)\xi_{m+n}\,, \qquad [\bar{\xi}_n, \bar{\xi}_m] = (n-m)\bar{\xi}_{m+n} \qquad n, m \in \mathbb{Z}.$$

  One may then use a more formal notation by denoting the generators by L_n:

$$[L_n, L_m] = (n-m)L_{m+n}\,, \quad [\bar{L}_n, \bar{L}_m] = (n-m)\bar{L}_{m+n}\,, \quad [L_n, \bar{L}_m] = 0\,, \quad n, m \in \mathbb{Z}. \tag{2.23}$$

- The above algebra is called the "Witt algebra".

NOTE: *Then* ξ_n *is a specific representation of the above algebra in* coordinate *basis. Conformal algebra may have other representations and/or some "internal" parts.*

▶▶ **Exercise 2.24:** *Show that the Witt algebra is a Lie algebra.*

- Although in 2d the conformal group is infinite dimensional one can still construct conformal invariant cross-ratios.
- In fact holomorphic and anti-holomorphic parts each has its own cross-ratios.

  ▶▶ **Exercise 2.25:** *Write out all conformal invariant cross-ratios for four and five points* $(z_i, \bar{z}_i)$, $i = 1, 2, 3, 4$ *or* 5.

  NOTE: *As we see the 2d conformal algebra has two copies, the left and right moving, Witt algebras.*

- $L_0, L_{\pm 1}$ generators of the Witt algebra form a subalgebra, because:

$$[L_0, L_\pm] = \mp L_\pm \,, \qquad [L_+, L_-] = 2L_0 \,.$$

  The above is nothing but $so(2,1)$ or equivalently (its double cover) $sl(2,R)$.

- The (1+1)d conformal algebra has hence a $sl(2,R)_L \times sl(2,R)_R \simeq so(2,2)$ subgroup.
- The 2d (Euclidean) conformal algebra is $so(3,1) \simeq sl(2,\mathbb{C})$. These transformation are generated by

$$f(z) = \frac{az+b}{cz+d}\,, \qquad ad - bc = 1\,, \qquad a, b, c, d \in \mathbb{C}\,. \tag{2.24}$$

  ▶▶ **Exercise 2.26:** *Show that two successive transformations as above indeed form a group produce by matrix multiplication of* 2×2 *matrix made out of* a, b, c, d.

- Generators of the Witt algebra in the coordinate basis are $L_+ = z^2\partial_z,\ L_0 = z\partial_z,\ L_- = \partial_z$.

  ▶▶ **Exercise 2.27:** *Write the generators of 2d Poincaré and dilatations in terms of* $sl(2,R)$ *generators.*

  ▶▶ **Exercise 2.28:** *Show that given* n *points with coordinates* $(z_1, z_2, \cdots, z_n)$, *using the* $SL(2,C)$ *transformations, can be brought to* $(0, 1, \infty, w_4, \cdots, w_n)$.
  Write explicitly the $SL(2,C)$ *transformations which does this.*

  ▶▶ **Exercise 2.29:** *Using the above write all the four and five point cross-ratios.*

  ▶▶ **Exercise 2.30:** *Gain intuition about conformal transformations* $z \to z^n$*: Use the 2d polar coordinates to study how the conformal map changes the shapes.*

  ▶▶ **Exercise 2.31:** *What are the scaling and special conformal transformations. Represent them as* $w = w(z)$.

NOTE: *Think what is the difference between 1+1 and 2d conformal algebras.*

▶▶ **Exercise 2.32:** *As discussed 2d conformal transformations are generated by all meromorphic diffeomorphism . These are not generically invertible if extended to the whole 2d complex plane. Show that* (2.24) *is the only globally invertible* ***holomorphic*** *map among conformal transformations.*

2.2.1 Representations of the 2d conformal group

- Since in this case the group is infinite dimensional the story will be different and representations are also infinite dimensional, in general.
- The states in the irreps of Witt algebra may be labeled by eigenvalues of L_0 (which is nothing but the scaling dimensions.).

  ▶▶ **Exercise 2.33:** *Discuss why L_0 is special.*

  ▶▶ **Exercise 2.34:** *Does Witt algebra admit second rank Casimirs?*

- As $d > 2$ case, one may define the notion of *quasi-primary* fields. In 2d, however, the *quasi-primary* fields are labels by eigenvalues of L_0 and $\bar{L}_0$, respectively the *holomorphic and anti-holomorphic* (or equivalently *left and right*) conformal weights $h, \bar{h}$:

$$\phi(z,\bar{z}) \to \phi'(w,\bar{w}) = \left(\frac{\partial w}{\partial z}\right)^{-h} \left(\frac{\partial \bar{w}}{\partial \bar{z}}\right)^{-\bar{h}} \phi(z,\bar{z})\,, \qquad z \to w = w(z)\,, \bar{z} \to \bar{w} = \bar{w}(\bar{z})\,. \tag{2.25}$$

  NOTE: *In the above we are considering $SL(2,C)$ transformations.*

  NOTE: *Under 2d rigid scaling the field has weight $\Delta = h + \bar{h}$ and under rotation eigenvalue (spin) $S = h - \bar{h}$.*

- ***Primary fields*** are a subset of quasi-primary fields which under *any* generic infinitesimal meromorphic diffeomorphism

$$z \to z + \xi(z)\,, \qquad \bar{z} \to \bar{z} + \bar{\xi}(\bar{z})\,,$$

  transform as

$$\phi(z,\bar{z}) \to \phi'(z,\bar{z}) = \phi(z,\bar{z}) + \delta\phi\,, \quad \delta\phi = -(h\partial\xi\ \phi + \xi\partial\phi) - (\bar{h}\bar{\partial}\bar{\xi}\ \phi + \bar{\xi}\bar{\partial}\phi)\,. \tag{2.26}$$

  NOTE: *The difference between primary and quasi-primary fields is that quasi-primary is defined only under the $SL(2,C)$ sector.*

▶▶ **Exercise 2.35:** *Given a primary field ϕ of conformal weights $(h, \bar{h})$, what is the conformal weights of $\partial\phi$ and $\bar{\partial}\phi$? What are their spin S and scaling dimension Δ?*

▶▶ **Exercise 2.36:** *Show that derivatives of primary field ϕ for generic $h, \bar{h}$ are NOT primaries. Are they quasi-primary? When these derivatives are primary?*

- Given a primary state (lowest weight state) of a 2d primary $|h, \bar{h}\rangle$ multiplet is built by acting Virasoro generators on $|h, \bar{h}\rangle$.
- Later on when we defined the conformal group as a symmetry of a QFT we will study another representation for the conformal algebra, based on the primary states and their (conformal) descendants.

3 Conformal group as symmetry of QFT's

- A generic Rel.QFT consists of fields in Irreps of Poincaré which are interacting with each other. These interactions are in general given by the local Lorentz invariant opt's of the theory.
- If we consider *free massless* fields the action is invariant under rigid scaling $x \to \lambda x$, IFF we assign appropriate scaling dimensions to the fields. For example, scalar fields in d dimensions have scaling dimension $(d-2)/2$, spinors $(d-1)/2$ and gauge fields with standard (canonical) kinetic term $(d-2)/2$.
- Then, a generic physical observables (local opt's) of the theory $\mathcal{O}$ will have a definite scaling dimension, $\Delta_{\mathcal{O}}$.
- If this operator is used to perturb/deform the theory the corresponding coupling will have dimension $d-\Delta_{\mathcal{O}}$; according to standard QFT terminology if this value is positive, zero or negative the operator is respectively called relevant, marginal and irrelevant opt.
- Invariance under rigid scaling is lost *if* we have a dimensionful coupling in the theory.

  ▶▶ **Exercise 3.1:** *Show the above explicitly.*

- Absence of dimensionful couplings does NOT imply scale invariance of the quantum theory because, couplings of QFT's generically have RG flow, running (with energy/scale). Therefore, even for marginal deformations coupling at different energies/scales will be different.
- According to standard Wilsonian picture, a QFT is defined around its **RG fixed point**. *At the fixed point, just by definition, QFT is scale invariant.* Perturbations around the fixed point may, however, not respect this symmetry.

- Therefore, the scaling symmetry is relevant to the study of QFT's.

  NOTE: *Scale invariance does not mean conformal invariance.*

- In what follows we would like to study
  - the restrictions scale/conformal invariance imposes on QFT to start with.
  - when is the scale/confromal symmetries exact (non-anomalous).
  - Starting from a conformal invariant fixed point, what this symmetry implies for physical observables (correlators) of the theory.
    NOTE: *This is essentially related to question of deformations/perturbations around the conformal fixed point.*

- As discussed 2d is special in some different ways:
  - scalar and gauge fields are dimensionless and hence we have more freedom in constructing CFT's.
  - The conformal algebra is infinite dimensional and hence more restrictive.

NOTE: *Therefore, 2d CFT is the most developed and studied among the CFT's. As before, we need to discuss CFT's in various dimensions, and in particular 2d separately. However, before that let us review some basic facts about symmetries in QFT's.*

3.1 Remarks on quantization, radial quantization

- There are some different ways to "quantize" a given field theory, including **canonical quantization** or **path integral** method.
- In the canonical method, we deal with operators and states and the *time evolution* of the system is given by the Hamiltonian, which is generator of time translations:

$$|\Phi(t)\rangle = U(t)|\Phi_{in}\rangle\,, \qquad U(t) = e^{i\mathbf{H}t}\,, \tag{3.1}$$

  NOTE: *In a field theory Hamiltonian is the generator of translation between constant time (spatial) slices.*

- We usually use perturbation theory and assume that $|\Phi_{in}\rangle$ is an approximate eigenstate of the "free" Hamiltonian.
- For a CFT, however, as discussed states/operators are labeled by eigenvalues of the Dilation operator D. It is hence, useful to quantize the theory in a way that uses this fact.

- This is achieved in the so-called **radial quantization**.
- Consider a Euclidean d dimensional space:

$$ds^2 = dx_0^2 + \sum_{i=1}^{d-1} dx_i^2 = dr^2 + r^2 d\Omega_{d-1}^2$$
$$= e^{2\rho}(d\rho^2 + R^2 d\Omega_{d-1}^2)\,, \qquad r = Re^\rho\,. \tag{3.2}$$

That is, R^d is conformal to $R \times S^{d-1}$.

▶▶ **Exercise 3.2:** *Show that translation in radial direction ρ corresponds to scaling in x_μ coordinates.*

- Therefore, a CFT on R^d is equivalent to a similar theory on d dimensional cylinder $C^d = R \times S^{d-1}$. This similar theory will have a conformal mass term.

▶▶ **Exercise 3.3:** *If we start with a scalar field theory on R^d with action*

$$S = -\int_{R^d} d^d x \, \left[\frac{1}{2}\partial_\mu \phi_a \partial^\mu \phi_a + V(\phi_a)\right]\,, \tag{3.3}$$

Then, show that under the conformal map to C^d the action becomes

$$S = -\int_{C^d} d^d x \, \left[\frac{1}{2}\nabla_\mu \phi_a \nabla^\mu \phi_a + \frac{1}{6}\mathcal{R}\phi_a^2 + V(\phi_a)\right]\,, \tag{3.4}$$

where $\frac{1}{6}\mathcal{R}\phi_a^2$ is called the conformal mass term and $\mathcal{R}$ is the Ricci scalar curvature of C^d.
Note that for $d = 2$ $C^2 = R \times S^1$ the space is flat while for $d > 2$ C^d is not flat and when moving from R^d to C^d for $d > 2$ we need to add the conformal mass term.

- One may use the ρ direction as the "time" to define the evolution. That is, R direction on C^d (the axis of the cylinder) would play the role of "time". This is called **radial quantization** of the field theory on R^d.
- The evolution operator in the radial quantization is hence:

$$|\Phi(\rho)\rangle = U(\rho)|\Phi_{in}\rangle\,, \qquad U(\rho) = e^{i\rho\mathbf{D}}\,, \tag{3.5}$$

NOTE: *In this case, the origin $r = 0$ corresponds to $\rho = -\infty$ where the $|\Phi_{in}\rangle$ state is defined.*

- Noting that Hamiltonian of radial quantization $\mathbf{D}$ is a unitary, the question of unitarity of the theory hence reduces to the question of unitarity or "unitarizability" of the representations of the $|\Phi_{in}\rangle$ states. Next, note that $|\Phi_{in}\rangle$ is defined at the origin $x_\mu = 0$ and this makes a direct connection with what we discussed earlier in the previous section, that to construct representations of conformal group we may only focus on the *internal* parts of conformal generators defined at $x = 0$.

- Recalling the operator state correspondence hereafter we use $|\mathcal{O}\rangle$ instead of $\mathcal{O}(0)$.
- As in the standard quantization, the "quasi-primary" states are simultaneously diagonalize D and Lorentz generators $J_{\mu\nu}$:

$$iD|\Delta, s\rangle = \Delta|\Delta, s\rangle\,, \qquad S^2|\Delta, s\rangle = s(s+d-3)|\Delta, s\rangle\,. \tag{3.6}$$

  NOTE: *The second relation above is written for integer s tensorial representation of $SO(d)$.*

- In the canonical quantization method, we assume the vacuum state $|0\rangle$ to be invariant under all conformal group transformations:

$$G|0\rangle = 0\,, \qquad \forall G \in \ conformal\ algebra\,.$$

- As always we have the standard operator-state correspondence, between *local* operators and states:

$$|\mathcal{O}\rangle \longleftrightarrow \mathcal{O}|0\rangle\,.$$

- The above mapping (3.2) was given for the Euclidean spaces. One may repeat similar mapping in the Minkowski case, with the following procedure:

  Wick rotate from $R^{d-1,1}$ to R^d, then do the conformal mapping to $R_\rho \times S^{d-1}$ and Wick rotate back along R_ρ by replacing $\tau = i\rho$.

- One may also use the standard path integral method which is somewhat "more covariant" than the canonical method. We will mainly consider that.

3.1.1 Norm and Unitarity in radial quantization

- A unitary QFT is a theory with positive definite norm on its Hilbert space, i.e.

$$\langle\Phi|\Phi\rangle \geq 0\,, \qquad \forall|\Phi\rangle\,, \quad \text{and} \quad \langle\Phi(t)|\Phi(t)\rangle \geq 0\,. \tag{3.7}$$

- In order the dynamics to preserve the norm, the Hamiltonian should be *Hermitian and bounded from below*, i.e. to have a positive definite spectrum.
- One may ask similar question about radial quantization and the spectrum of dilation operator:

  What is the condition for unitarity in the radial quantization?

- Let us focus on the Euclidean case, where the conformal group is $SO(d+1,1)$. Then, recalling that

$$P_\mu = L_{-1,\mu} + L_{d+1,\mu}\,, \qquad K_\mu = -L_{-1,\mu} + L_{d+1,\mu}\,, \qquad D \equiv iL_{-1,d+1}\,, \tag{3.8}$$

and that

$$L^\dagger_{-1,\mu} = -L_{-1,\mu}\,, \qquad L^\dagger_{d+1,\mu} = L_{d+1,\mu}\,,$$

we learn that

$$(P_\mu|\Phi\rangle)^\dagger = \langle\Phi|K_\mu\,, \qquad (K_\mu|\Phi\rangle)^\dagger = \langle\Phi|P_\mu\,. \tag{3.9}$$

- Next we recall that

$$[K_\mu, K_\nu] = 0\,, \qquad [P_\mu, P_\nu] = 0\,, \tag{3.10}$$

$$[K_\mu, D] = +K_\mu\,, \qquad [P_\mu, D] = -P_\mu\,, \tag{3.11}$$

$$[D, L_{\mu\nu}] = 0\,, \qquad [P_\mu, K_\nu] = 2(D\delta_{\mu\nu} + iL_{\mu\nu})\,. \tag{3.12}$$

- The states/operators are labeled by eigenvalues of iD, the scaling dimension denoted by Δ and the representations of the $SO(d)$ Lorentz group (whose generators are $L_{\mu\nu}$). The latter is nothing but the spin representation of $SO(d)$. Explicitly, a generic state is $|\Delta, \{s_i\}\rangle$, $i = 1, \cdots, [d/2]$:

$$iD|\Delta, \{s_i\}\rangle = \Delta|\Delta, \{s_i\}\rangle\,. \tag{3.13}$$

- A given $SO(d)$ spin state itself, can be represented by the highest weight states $h_i\,, i = 1, \cdots, [d/2]$.

- In this notation, P_μ **raises** the scaling dimension and K_μ **lowers** it:

$$iD(P_\mu|\Delta, \{s_i\}\rangle) = (\Delta+1)(P_\mu|\Delta, \{s_i\}\rangle)\,, \qquad iD(K_\mu|\Delta, \{s_i\}\rangle) = (\Delta-1)(K_\mu|\Delta, \{s_i\}\rangle)\,. \tag{3.14}$$

►► **Exercise 3.4:** *Show the above using* (3.10).

- Conformal primaries denoted by $|\Delta_0, \{s_i\}\rangle$, as defined before, are the states which are killed by K_μ:

$$K_\mu|\Delta_0, \{s_i\}\rangle = 0\,, \tag{3.15}$$

- Given any primary operator/state, one can construct a conformal multiplet/representation, based on this state by acting with all conformal generators on the primary state. These other states are *descendents* of the primary.

►► **Exercise 3.5:** *If a primary has scaling dimension* Δ_0*, show that the corresponding descendents have scaling dimension* $\Delta_0 + n\,, n \in \mathbb{N}$.

- The primaries are hence states with lowest scaling dimension in a given conformal multiplet.

- The conformal multiplets are generically infinite dimensional, because one can consider descendents of the form $P_{\mu_1}P_{\mu_2}\cdots P_{\mu_n}|\Delta_0, \{s_i\}\rangle$ for arbitrarily large n.

3.1.2 Unitary bounds

- We can now invoke the unitarity conditions (3.7) on a primary state or its descendents and read off the condition on Δ_0. Depending on which level of descendents we consider we will obtain various inequalities, e.g. up to level two

$$\langle \Delta_0, \{s_i\} | \Delta_0, \{s_i\} \rangle \geq 0 \,, \tag{3.16}$$
$$||\zeta \cdot P | \Delta_0, \{s_i\} \rangle ||^2 \geq 0 \,, \tag{3.17}$$
$$||\xi \cdot P \zeta \cdot P | \Delta_0, \{s_i\} \rangle ||^2 \geq 0 \,. \tag{3.18}$$

- Let us explore the level one condition. Using $P_\mu^\dagger = K_\mu$ and the $[P_\mu, K_\nu]$ commutation in (3.10) and that $|\Delta_0, \{s_i\}\rangle$ is a primary state we learn that

$$\zeta_\mu^* \zeta_\nu \langle \Delta_0, \{s_i\} | (\Delta_0 \delta_{\mu\nu} - iL_{\mu\nu}) | \Delta_0, \{s_i\} \rangle \geq 0 \,, \tag{3.19}$$

for any given ζ_μ.

- $iL_{\mu\nu}$ is an operator which has $SO(d)$ vector indices, as well as acting on $SO(d)$ spin state $|\{s_i\}\rangle$. Therefore, it is a product of $L = 1$ and $L = S$ SO(d) generators, explicitly:

$$\begin{aligned} iL_{\mu\nu} &= V_{\mu\nu}^{ab} S_{ab} \,, \qquad V_{\mu\nu}^{ab} = i(\delta_\mu^a \delta_\nu^b - \delta_\nu^a \delta_\mu^b) \,, \\ &\equiv L \cdot S \\ &= \frac{1}{2}(-L^2 - S^2 + (L+S)^2) \,. \end{aligned} \tag{3.20}$$

where $a, b = 1, \cdots d$.

- Given a spin state S, we hence need to decompose $1 \bigotimes S$ and compute $(L+S)^2$ for this state.

- The BIGGEST eigenvalue of this decompose spin states, denoted by $\ell_{\mathbf{s}}$, will determine the bound on Δ_0:

$$\Delta_0 \geq \ell_{\mathbf{s}} \,. \tag{3.21}$$

- The rest is just a group and representation theory of $SO(d)$ to arrive at the largest value of $\ell_{\mathbf{s}}$ and obtain the bound. Here we just summarize the result. The details may be found in `hep-th/9712074` and references therein.

- One can show that the above "level 1" analysis yields the unitarity bound on *non-scalar* states. For scalar (spin zero) states the level 1 analysis gives $\Delta_0 \geq 0$. One may then consider the "level 2" analysis for scalars to find

$$\Delta_0 (\Delta_0 - \frac{d-2}{2}) \geq 0 \,.$$

For non-scalar states level 2 or higher analysis does not yield any new condition.

►► **Exercise 3.6:** *Work out the above.*

- **Summary of the unitarity bounds in dimension d:**

$$\textbf{Scalar}: \qquad \Delta_0 \geq \frac{d-2}{2}\,, \tag{3.22}$$

$$\textbf{Spin 1/2}: \qquad \Delta_0 \geq \frac{d-1}{2}\,, \tag{3.23}$$

$$\textbf{Vector}: \qquad \Delta_0 \geq d-1\,, \tag{3.24}$$

$$\textbf{Antisymmetric } F_{\mu\nu}: \qquad \Delta_0 \geq d/2\,. \tag{3.25}$$

The last item in the above corresponds to the field strength of a vector gauge field.

▶▶ **Exercise 3.7:** *Work out the details of the analysis leading to the above bounds.*

▶▶ **Exercise 3.8:** *If we denote the vector state* ***saturating*** *the unitarity bound by $|J^\mu\rangle$, show that $P_\mu|J^\mu\rangle = 0$. This establishes the fact that conformal "primary currents" are conserved and will couple to gauge fields.*

▶▶ **Exercise 3.9:** *Show that FREE fields in any dimension saturate the above bounds.*

NOTE: *Note that our group theory analysis is smart enough to distinguish the fact that the "vector" field by itself is not a physical state and we need to consider gauge invariant combinations like field strength. That is, the physical vectors does not correspond to gauge fields A_μ, but to their currents.*

▶▶ **Exercise 3.10:** *As argued in the previous section states of given scaling dimension cannot be eigenstate of P^2, unless we are dealing with massless states, $P^2|\Delta_0, \{s_i\}\rangle = 0$. Convince yourself that this statement is true for all descendents too. (Note that only for free fields P^2 is the "mass".*

▶▶ **Exercise 3.11:** *Given the above exercise, convince yourself that we may always deal with representations of the Lorentz little group associated with "massless" states, $SO(d-2)$.*

NOTE: *The above unitarity bounds* (3.22) *is nothing but the BF bound for mass of states on an AdS_{d+1} background. This is necessitated by the AdS/CFT; to come later....*

- **Unitarity Bounds in the 2d case:** In the 2d case the states are labeled by $SL(2,R)\times SL(2,R)$ representations, which in turn are labeled by eigenvalues of L_0 and $\bar{L}_0$.

  ▶▶ **Exercise 3.12:** *If we denote the eigenvalues of L_0 and $\bar{L}_0$ by h and $\bar{h}$, respectively, show that the unitarity bound is*

$$h \geq 0\,, \qquad \bar{h} \geq 0\,.$$

where in terms of $so(2,2)$ or $so(3,1)$ quantum numbers, scaling dimension Δ and spin S, these are

$$\Delta = h + \bar{h}\,, \qquad S = h - \bar{h}\,.$$

- We will return to 2d conformal algebra representations later on when we introduce the notion of central charge and Virasoro algebra. The central charge is not a geometrical notion and is a characteristic of the CFT in question.

3.2 Classical Noether currents for conformal symmetry

- Continuous global symmetries,

$$\boxed{\phi(x) \to \phi'(x') = \Phi(\phi(x))\,, \quad x \to x'}\,, \qquad S[\phi] = \int d^d x \mathcal{L}(\phi, \partial\phi, \cdots; x) = S[\phi']\,, \tag{3.26}$$

  lead to conserved Noether currents.

- To compute the Noether current consider *infinitesimal* transformations

$$\boxed{x'^\mu = x^\mu + \epsilon_a \frac{\delta x^\mu}{\delta \epsilon_a}\,, \qquad \phi'(x') = \phi(x) + \epsilon_a \frac{\delta \Phi}{\delta \epsilon_a}\,.} \tag{3.27}$$

  ►► **Exercise 3.13:** *Compute* $\delta\phi \equiv \phi'(x) - \phi(x)$.

- Invariance of action then leads to $\partial_\mu J^\mu_a = 0$, where

$$J^\mu_a = \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L} \right) \frac{\delta x^\nu}{\delta \epsilon_a} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \frac{\delta \Phi}{\delta \epsilon_a}\,. \tag{3.28}$$

  ►► **Exercise 3.14:** *Work out the above.*

- Given a conserved current one can define conserved charges Q_a

$$Q_a = \int d^{d-1} x J^0_a\,. \tag{3.29}$$

  NOTE: *One may define the conserved current up to the divergence of a two-form.*

- **Conserved currents for conformal group in** $d > 2$**:**

  ►► **Exercise 3.15:** *For a CFT with primary fields* ϕ_i*:*

  I. *Work out the* Noether current for space-time translations, the energy-momentum tensor $T_{\mu\nu}$.

II. *Show that the* ***Noether current for Lorentz transformations***

$$\mathcal{L}_{\alpha\mu\nu} = T_{\alpha\mu}x_\nu - T_{\alpha\nu}x_\mu\,, \tag{3.30}$$

if $T_{\mu\nu}$ ***is symmetric****. Note that the angular momentum, may have an internal* ***spin*** *part* $\mathbf{S}_{\alpha\mu\nu}$ *and the "total angular momentum" should be conserved.*

III. *Show that the conserved* ***Noether current of rigid scaling*** *is*

$$J_\mu = T_{\mu\nu}x^\nu, \tag{3.31}$$

where $T_{\mu\nu}$ *is the energy momentum tensor and defined as*

$$T_{\mu\nu} = -\frac{2}{\sqrt{g}}\frac{\delta S}{\delta g^{\mu\nu}}\,.$$

Show that conservation of J_μ *(i.e. scale invariance) implies* $T^\mu{}_\mu = 0$*, i.e.* ***tracelessness of energy momentum tensor****.*

IV. *Show that the* ***Noether current of special conformal transformations*** *is*

$$K_{\mu\nu} = T_{\mu\alpha}(2x_\alpha x_\nu - x^2\delta_{\alpha\nu})\,. \tag{3.32}$$

V. *Show that* $K_{\mu\nu}$ *is conserved if energy momentum conservation, Lorentz and scaling invariance holds.*

HINT: For the above exercises note the definition of the conformal transformation (2.4) and recall that $T_{\mu\nu} \equiv -\frac{2}{\sqrt{g}}\frac{\delta S}{\delta g^{\mu\nu}}$. Then, note that since we are dealing with a diffeomorphism, the conserved Noether current and charge for the conformal transformations is

$$J^\mu_{conformal} = T^{\mu\nu}\xi_\nu\,, \qquad Q_\xi = \int d^3x\; T^{0\nu}\xi_\nu\,, \tag{3.33}$$

where for our case ξ_ν is given in (2.8).

NOTE: *As implied by* ***V.*** *part of the above exercise, it seems that, at least at classical level and when we ignore the internal parts of conformal generators (like spin and* Δ*), Poincaré and scaling invariance is enough for full conformal invariance. Whether this result continues to be true when we consider the internal parts and quantum effects has prompted many researches and studies in the literature. We will briefly discuss this later.*

3.3 Ward identities for conformal symmetry

- The notion of symmetry, Noether current and charge defined above are classical. One may wonder if they remain at quantum level.

- There are some different and somewhat complementary ways to explore the above question, e.g.

I. noting that all quantum effects are encoded in the (Wilsonian) effective action, the question of *anomaly* reduces to the invariance of effective action under the symmetry transformations (or perhaps a quantum corrected transformation).

II. OR, recall that when a transformation (3.27) is a symmetry of a theory, all n-point functions should be invariant under that transformation

$$\boxed{\langle \mathcal{O}_1(x_1')\mathcal{O}_2(x_2')\cdots\mathcal{O}_n(x_n')\rangle = \langle \tilde{\mathcal{O}}_1(x_1)\tilde{\mathcal{O}}_2(x_2)\cdots\tilde{\mathcal{O}}_n(x_n)\rangle\,,} \tag{3.34}$$

where

$$\mathcal{O}_i(x') = \mathcal{O}_i[\phi(x')]\,,\qquad \tilde{\mathcal{O}}(x) \equiv \mathcal{O}[\Phi(\phi(x))]\,, \tag{3.35}$$

(operatorial form of $\mathcal{O}_i$ are not necessarily the same).

▶▶ **Exercise 3.16:** *Show* (3.34).
HINT: To show the above note that in the path integral we are integrating over ALL field configurations and ϕ is the dummy variable inside path integral. What assumption about the path integral measure should be made?!

One may expand (3.34), recalling that

$$S[\phi'] = S[\phi] + \int d^dx\partial_\mu J^\mu_a\epsilon_a(x)\,,\qquad D\phi = D\phi'\,. \tag{3.36}$$

NOTE: *As in the Noether theorem we are assuming $\epsilon_a = \epsilon_a(x)$ and will relax it later.*

NOTE: *The second equation above is an assumption.*

The RHS of (3.34) may now be expanded for infinitesimal symmetry generating transformations to obtain:

$$\epsilon_a(x)\langle\delta_a(\prod \mathcal{O})\rangle = \int d^dx\ \partial_\mu\langle J^\mu_a(x)\mathcal{O}_1\mathcal{O}_2\cdots\mathcal{O}_n\rangle\epsilon_a(x)\,, \tag{3.37}$$

where

$$\epsilon_a(x)\delta_a(\prod \mathcal{O}) \equiv \mathcal{O}_1[\phi'(x_1)]\mathcal{O}_2[\phi'(x_2)]\cdots\mathcal{O}_n[\phi'(x_n)] - \mathcal{O}_1[\phi(x_1)]\mathcal{O}_2[\phi(x_2)]\cdots\mathcal{O}_n[\phi(x_n)].$$

OR

$$\delta_a(\prod \mathcal{O}) = \sum_{i=1}^n \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\left(\frac{\delta\mathcal{O}_i}{\delta\phi}\delta_a\phi\right)\Big|_{x=x_i}\cdots\mathcal{O}_n(x_n). \tag{3.38}$$

NOTE: *Eq.(3.37) should be true for all $\epsilon_a(x)$. And hence one may drop ϵ_a from both sides and formally obtain:*

$$\langle\delta_a(\prod \mathcal{O})\rangle = \partial_\mu\langle J^\mu_a(x)\mathcal{O}_1\mathcal{O}_2\cdots\mathcal{O}_n\rangle\,, \tag{3.39}$$

▶▶ **Exercise 3.17:** *Work out and simplify* (3.37) *for free field theory and when* $\mathcal{O}_i = \phi(x_i)$ *and for linear global transformations.*

▶▶ **Exercise 3.18:** *Integrate both sides of the above equation over* x *coordinate and simplify the equation. This equation may also be called Ward identity.*

III. OR, one may directly explore whether the conservation of Noether current $\partial_\mu J^\mu_a = 0$ holds IF J_μ is viewed as a quantum operator. This operator equation holds IF

$$\langle \partial_\mu J^\mu_a(x)\, X(x_1, x_2, \cdots, x_n)\rangle = \partial_\mu \langle J^\mu_a(x)\, X(x_1, x_2, \cdots, x_n)\rangle \stackrel{?}{=} 0 \quad \forall X\,, x_i \neq x\,. \tag{3.40}$$

NOTE: *It is important to remember that* $x \neq x_i$*, otherwise the above is not strictly true. This is to avoid operator ordering issues in the local QFT's.*

▶▶ **Exercise 3.19:** *Show that* (3.40) *and* (3.34) *are basically equivalent.*

The above is the Ward identity in a compact and general form.

▶▶ **Exercise 3.20:** *Work out the above Ward identity for a free field theory and for* $X = \phi(x_1)\phi(x_2)\cdots\phi(x_n)$ *and when for linear transformations* $\delta\phi = \epsilon_a G_a \phi$.

3.3.1 Ward identities for conformal group in $d > 2$

- The above arguments was for a general symmetry, we may now restrict our attention to the conformal group. We hence start with the currents worked out in exercise **??** and consider translations, Lorentz transformation, dilation and special conformal transformations separately.

  ▶▶ **Exercise 3.21:** ***Ward identity for energy-momentum conservation:*** *Work out and simplify the Ward identity* (3.39) *and/or* (3.40) *for the energy-momentum tensor which is the conserved current for spacetime translations. Note that for this case* $\delta\mathcal{O} = -\epsilon^\mu \partial_\mu \mathcal{O}$.
  Answer:

$$\partial_\mu \langle T^{\mu\nu}(x)\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)\rangle = -\sum_{i=1}^{n} \delta(x - x_i)\frac{\partial}{\partial x^\nu_i}\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)\rangle\,. \tag{3.41}$$

▶▶ **Exercise 3.22:** *Suppose that we have a QFT with partition function* $\mathcal{Z}$ *and generating functional* $\mathcal{W}$*:*

$$\mathcal{Z}[g] = \int [D\phi]_g\, e^{-S[\phi;g]}\,, \qquad \mathcal{W} = -\ln \mathcal{Z}\,,$$

where g is the background metric which is treated as a "classical" field. Show that

$$\langle T^{\mu\nu}\rangle = -\frac{2}{\sqrt{g}}\frac{\delta\mathcal{W}[g]}{\delta g^{\mu\nu}}\,. \tag{3.42}$$

Discuss on metric dependence of the measure $[D\phi]_g$.

▶▶ **Exercise 3.23:** ***Ward Identity for angular momentum conservation:*** *If*

$$\delta\mathcal{O} = -\epsilon^{\mu\nu}[(x_\mu\partial_\nu - x_\nu\partial_\mu) + \mathbf{S}_{\mu\nu}]\mathcal{O}$$

then work out the angular momentum ward identity

$$\langle T^{[\mu\nu]}(x)\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)\rangle = -\sum_{i=1}^{n}\delta(x-x_i)\mathbf{S}_i^{\mu\nu}\langle\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)\rangle\,, \tag{3.43}$$

where $\mathbf{S}_i^{\mu\nu}$ is the spin of operator $\mathcal{O}_i$ and $T^{[\mu\nu]} = T^{\mu\nu} - T^{\nu\mu}$ is the antisymmetric part of T.

▶▶ **Exercise 3.24:** ***Ward Identity for rigid scaling (dilation) symmetry:*** *For operators of given scaling dimension:*

$$\delta\mathcal{O}_i = -(x^\mu\partial_\mu + \Delta_i)\mathcal{O}_i\,,$$

show that the Ward identity becomes

$$\langle T^\mu_\mu(x)\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)\rangle = -\sum_{i=1}^{n}\delta(x-x_i)\Delta_i\langle\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)\rangle\,, \tag{3.44}$$

where in the above (3.41) *has been used.*

NOTE: *The scale invariance, like other continuous symmetries, may be anomalous. One way to check this is to compute ene.mom. tensor using effective action. We will return to scaling anomaly, usually called "trace anomaly" later.*

NOTE: *If* (3.44) *is not satisfied for all $\mathcal{O}_i$ then scaling symmetry is anomalous; we have trace anomaly.*

▶▶ **Exercise 3.25:** *Work out the Ward identity for special conformal transformations. Does it reduce to scaling and Poincaré invariance Ward identities?*

NOTE: *The answer to the above together with* (3.41) *and* (3.44) *provide Ward identities for the conformal symmetry.*

- One may try to repeat the above in the canonical language and in terms of operators, rather than path integral. In this case the Noether charge Q_a turns to an operator defined on the QFT Hilbert space. The field transformation in this case can be written as
$$\delta_a \Phi = \frac{\delta \Phi'(x')}{\delta \epsilon_a} = i[Q_a, \Phi]\,. \tag{3.45}$$

  ►► **Exercise 3.26:** *Show the above.*

3.4 Conformal invariance and QFT correlation functions, $d > 2$

- Let us suppose that we have a conformally invariant theory and $\mathcal{O}_i$ be its quasi-primary operators of scaling dimension Δ_i.

  ►► **Exercise 3.27:** *Show that $\partial_\mu \mathcal{O}_i$ is an operator with scaling dimension $\Delta_i + 1$, while not necessarily a quasi-primary operator.*

- Under a general conformal transformations $x \to x' = x'(x)$,
$$\mathcal{O}_i(x) \to \mathcal{O}'_i(x') = \left|\frac{\partial x'}{\partial x}\right|^{-\Delta_i} \mathcal{O}_i(x)\,. \tag{3.46}$$

  If operators $\mathcal{O}_1$ and $\mathcal{O}_2$ are **Lorentz scalars**, the translation invariance implies
$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle = f(|x_1 - x_2|)\,,$$
  where f is a scalar function.

  ►► **Exercise 3.28:** *Similar result also holds for non-scalar operators. Argue that $\langle \mathcal{O}_1\mathcal{O}_2\rangle$ is in general non-zero IFF $\mathcal{O}_1$ and $\mathcal{O}_2$ are in the same Lorentz Rep.*
  Work out the most general form of two point function when $\mathcal{O}$ are spinor or vector. For a reference discussing this, e.g. see `H. Osborn and A. C. Petkou, Annals Phys. 231, 311 (1994) [arXiv:hep-th/9307010].`

  Then invariance under rigid scaling implies that
$$\langle \mathcal{O}_1(\lambda x_1)\mathcal{O}_2(\lambda x_2)\rangle = \lambda^{-\Delta_1+\Delta_2}\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle$$
  and hence
$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle = \frac{C_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}}\,. \tag{3.47}$$
  Next, we use invariance under special conformal transformations:
$$\langle \mathcal{O}_1(x'_1)\mathcal{O}_2(x'_2)\rangle = \left|\frac{\partial x'}{\partial x}\right|^{-\Delta_1/d}_{x=x_1} \left|\frac{\partial x'}{\partial x}\right|^{-\Delta_2/d}_{x=x_2} \langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle \tag{3.48}$$

where

$$\left|\frac{\partial x'}{\partial x}\right| = \frac{1}{(1 - 2b\cdot x + b^2x^2)^d}\,.$$

Therefore,

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle = \begin{cases} \frac{N_{12}}{|x_1-x_2|^{2\Delta_1}} & \Delta_1 = \Delta_2 \\ \\ 0 & \Delta_1 \neq \Delta_2 \end{cases} \tag{3.49}$$

NOTE: *Scale invariance does not imply* $\Delta_1 = \Delta_2$ *in* (3.47).

►► **Exercise 3.29:** *Complete the steps of the above computation.*

NOTE: *The above analysis is for the Euclidean theory. In the Lorentzian signature* (**??**) *is rewritten as*

$$\langle T(\mathcal{O}_1(x_1)\mathcal{O}_2(x_2))\rangle = \begin{cases} \frac{N_{12}}{|(x_1-x_2)^2+i\epsilon|^{\Delta_1}} & \Delta_1 = \Delta_2 \\ \\ 0 & \Delta_1 \neq \Delta_2 \end{cases}$$

- Similarly one can work out the three point function:

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \frac{C_{123}}{|x_1-x_2|^c|x_1-x_3|^b|x_2-x_3|^a}\,,$$

where scale invariance implies $a+b+c = \Delta_1+\Delta_2+\Delta_3$.
Invariance under special conformal transformations leads to

$$b+c = 2\Delta_1, \quad a+c = 2\Delta_2, \quad a+b = 2\Delta_3.$$

►► **Exercise 3.30:** *Work out the above.*

Finally, we obtain

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \frac{C_{123}}{|x_1-x_2|^{\Delta_1+\Delta_2-\Delta_3}|x_1-x_3|^{\Delta_1+\Delta_3-\Delta_2}|x_2-x_3|^{\Delta_2+\Delta_3-\Delta_1}}\,. \tag{3.50}$$

- The spacetime dependence of higher point functions is not fixed by conformal invariance, because we have conformal invariant *cross-ratios.*
For these cases the spacetime dependence is hence fixed to be functions of these cross-ratios.

►► **Exercise 3.31:** *Show that the four point function is restricted to*

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle = F(R_1,R_2)\prod_{i<j}^{4}|x_i-x_j|^{\Delta/3-\Delta_i-\Delta_j}\,, \tag{3.51}$$

where
$$R_1 = \frac{|x_1 - x_2||x_3 - x_4|}{|x_1 - x_3||x_2 - x_4|}, \qquad R_2 = \frac{|x_1 - x_2||x_3 - x_4|}{|x_2 - x_3||x_1 - x_4|},$$
and $\Delta = \sum_{i=1}^{4} \Delta_i$.

3.4.1 Scale vs. conformal invariance

- One of the questions which has been asked but is still an open question in general, is the relation between scale and conformal invariance in QFT's.
- This question is interesting because QFTs at RG fixed points are scale invariant and one would naturally like to know if they are also conformal invariant.
- As discussed earlier (*cf.* exercise 3.14) at classical level conservation of the Noether conserved current associated with special conformal transformations follows from the Poincaré and scale invariance. That is, classically scale+Poincaré invariance results in conformal invariance.

  ►► **Exercise 3.32:** *Does Ward identity for special conformal transformations also follow from the Ward identities of Poincaré +scaling, i.e.* (3.41), (3.43) *and* (3.44)*?*

- On the other hand, as discussed above spacetime dependence of correlation functions, including 2pt or 3pt functions is fixed once we use full conformal invariance and not just scaling.
- So one may wonder whether scale +Poincaré invariance implies conformal invariance.
- In the 2d case, we have the Zamolodchikov-Polchinski theorem proving that any 2d QFT with properties
  - **Unitarity**,
  - **Poincaré invariance (relativistic causality)**,
  - **discrete spectrum in scaling dimension**,
  - **unbroken scale invariance**,

  is necessarily a CFT.

  `Ref: A. B. Zamolodchikov, JETP Lett. 43 (1986) 730 [Pisma Zh. Eksp. Teor. Fiz. 43 (1986) 565].`
  `J. Polchinski, Nucl. Phys. B 303, 226 (1988).`

- In the context of 2d and 4d QFT's it has been shown that `[Y. Nakayama, arXiv:1302.0884]`:

  ....As of January 2013, our consensus is that there is no known example of scale invariant but non-conformal field theories in d=4 under the above mentioned five assumptions...

- One may relax either of the above five assumptions and try to construct examples of scale and Poincaré invariant QFTs which are NOT CFTs.
- Such examples in 2d has been studied.
- 3d example: *Wilson-Fisher fixed point and conformal invariance of 3d Ising?!*
`K. G. Wilson and M. E. Fisher, Phys. Rev. Lett. 28, 240 (1972);`
`S. Rychkov, arXiv:1111.2115 [hep-th].`
- See `[arXiv:1302.0884]` for discussions and examples.

4 2d conformal symmetry and 2d CFTs

- So far we mainly discussed conformal group and its representations in $d > 2$ dimensions. In the 2d case we have the peculiar feature that the conformal group is infinite dimensional.
- The conformal group in 2d is generated by two copies of meromorphic coordinate transformations and their generators satisfy two copies of the Witt algebra, one for right movers and one for left movers.
- Witt algebra has an $SL(2, R)$ subgroup, and hence 2d conformal group has $SL(2, R) \times SL(2, R) \simeq SO(3, 1)$ subalgebra.
- Witt algebra admits a *central extension*. This centrally extended Witt algebra is called *Virasoro* algebra:
$$[L_n, L_m] = (n - m)L_{m+n} + \frac{\mathbf{c}}{12} n(n^2 - 1)\delta_{m+n} \,. \tag{4.1}$$
$\mathbf{c}$ is the central charge.
- The central charge $\mathbf{c}$, unlike the L_n's do not have a direct coordinate interpretation; it is a property of the 2d CFT.
- A generic 2d CFT is hence invariant under two copies of Virasoro algebra. The associated left and right central charges $\mathbf{c}$ and $\bar{\mathbf{c}}$ can be different.

NOTE: *The central charge* $\mathbf{c}$ *does not appear in the* $SL(2, R)$ *part of Virasoro generated by* $L_0, L_\pm$.

►► **Exercise 4.1:** *Show under*
$$L_n \to \mathcal{L}_n = L_n + x_n \,, \tag{4.2}$$
where x_n *are some commuting operators. The the* $[L_n, x_m]$ *commutators such that the algebra of* $\mathcal{L}_n$ *reduces to the Witt algebra.*
Hint: *Use the ansatz* $[L_n, x_m] = Ax_{m+n} + B\delta_{m+n}$ *and find* A, B *coefficients.*

- We will return to the physical meaning of the central charge $\mathbf{c}$ in section 4.6.1.

4.1 Unitary representations of the Virasoro algebra

- In order to construct unitary reps we need to first construct a normalized *vacuum state* $|0\rangle$. This state must be killed by the "global" part of the conformal group $SL(2,C)$. This is the "invertible" part of the 2d conformal algebra, the Virasoro algebra. Explicitly,

$$L_0|0\rangle = 0\,,\ L_{\pm}|0\rangle = 0\,,\qquad \bar{L}_0|0\rangle = 0\,, \bar{L}_{\pm}|0\rangle = 0\,. \tag{4.3}$$

- Next, we note that

$$L_{-n} = L_n^\dagger\,, \tag{4.4}$$

►► **Exercise 4.2:** *Show the above using the Virasoro algebra.*
and hence $L_{-n}\,, n > 0$ acts as a **raising** operator while $L_n\,, n > 0$ as **lowering** operator. We therefore, demand

$$L_n|0\rangle = 0 \qquad n \geq -1\,. \tag{4.5}$$

NOTE: *Compared to the $d > 2$ case, $L_n\,, n > 0$ are all lowering operators like the K_μ.*

- If we demand non-negative norm for all $L_{-n}|0\rangle$ states, we find

$$||L_{-n}|0\rangle||^2 = \frac{\mathbf{c}}{12}n(n^2-1) \geq 0\,. \tag{4.6}$$

- So, unitarity of the CFT requires

$$\boxed{\mathbf{c} \geq 0\,, \qquad \bar{\mathbf{c}} \geq 0\,,} \tag{4.7}$$

and If $\mathbf{c} = 0$ then $||L_{-n}|0\rangle|| = 0$.

- After defining the vacuum state, we can construct the primary states. To this end, let us recall the operator-state correspondence: a primary state is the state generated by the action of a local primary operator of conformal weights $(h, \bar{h})$ on the vacuum, i.e.

$$|h, \bar{h}\rangle = \mathcal{O}(z = 0)|0\rangle.$$

- Using the definition of a primary operator and its behavior under conformal transformation (2.26):

$$\delta\mathcal{O} = -(h\partial\xi\ \mathcal{O} + \xi\partial\mathcal{O}) - (\bar{h}\bar{\partial}\bar{\xi}\ \mathcal{O} + \bar{\xi}\bar{\partial}\mathcal{O})\,, \tag{4.8}$$

and (2.22) and (3.45) we obtain

$$\begin{aligned} [L_n, \mathcal{O}(z,\bar{z})] &= h(n+1)z^n\mathcal{O}(z,\bar{z}) + z^{n+1}\partial\mathcal{O}(z,\bar{z})\,,\\ [\bar{L}_n, \mathcal{O}(z,\bar{z})] &= \bar{h}(n+1)\bar{z}^n\mathcal{O}(z,\bar{z}) + \bar{z}^{n+1}\bar{\partial}\mathcal{O}(z,\bar{z})\,, \end{aligned} \tag{4.9}$$

for $n \geq -1$.
►► **Exercise 4.3:** *Show the above.*

- Finally we learn that:

$$
\begin{aligned}
L_0|h,\bar{h}\rangle = h|h,\bar{h}\rangle\,, \qquad \bar{L}_0|h,\bar{h}\rangle = \bar{h}|h,\bar{h}\rangle\,, \\
L_n|h,\bar{h}\rangle = 0\,, \qquad \bar{L}_n|h,\bar{h}\rangle = 0\,, \qquad n>0\,.
\end{aligned}
\tag{4.10}
$$

The above defines the primary states.
NOTE: *Once again $L_n\,, n>0$ acts as the lowering operator and the primary is the state which is killed by the lowering operators; it is the lowest weight states in its "family".*

►► **Exercise 4.4:** *Show the unitarity condition for any primary state $|h,\bar{h}\rangle$ is $h\,,\bar{h}\geq 0$.*

- **The Verma module:**
  - As pointed out above number of lowering and raising operators in the 2d conformal algebra is infinite. Therefore, there are many different ways to construct descendents of a given primary state. The descendents may be labeled by their conformal weights, e.g.

$$
|\{n_i\};h,\bar{h}\rangle \equiv \prod_i L_{-n_i}|h,\bar{h}\rangle\,, \qquad \sum n_i = N\,, \tag{4.11}
$$

    has conformal weight $N+h$.
  - All the states of the form (4.11) are level N states in the Verma module based on $|h,\bar{h}\rangle$.

►► **Exercise 4.5:** *Compute how many descendents at level N are there.*

►► **Exercise 4.6:** *Show that if two primary states are orthogonal to each other, all the states in their Verma modules are also orthogonal:*

$$
\langle h'|h\rangle = 0 \quad \Rightarrow \quad \langle\{k_i\},h'|\{p_i\},h\rangle = 0\,. \tag{4.12}
$$

4.2 Correlation functions in 2d CFTs

NOTE: *Hereafter we will be only considering the **Euclidian** case.*

►► **Exercise 4.7:** *Show that in this case, $S = h-\bar{h}\in\mathbb{Z}$, while $\Delta = h+\bar{h}$ can be an arbitrary positive real number.*

- The above arguments may directly be extended to 2d CFTs. In 2d (quasi)primary fields/operators are specified by their conformal weights $(h,\bar{h})$.

- Suppose that we have primary fields/operators $\mathcal{O}_i = \mathcal{O}_i(z,\bar{z})$ with weights $(h_i, \bar{h}_i)$, and consider conformal transformations $z \to w = w(z)\,, \bar{z} \to \bar{w} = \bar{w}(\bar{z})$. We then have

$$\langle \tilde{\mathcal{O}}_1(w_1,\bar{w}_1)\tilde{\mathcal{O}}_2(w_2,\bar{w}_2)\cdots\tilde{\mathcal{O}}_n(w_n,\bar{w}_n)\rangle = \prod_{i=1}^{n} \left|\frac{dw}{dz}\right|_{w=w_i}^{-h_i} \left|\frac{d\bar{w}}{d\bar{z}}\right|_{\bar{w}=\bar{w}_i}^{-\bar{h}_i} \langle \mathcal{O}_1(z_1,\bar{z}_1)\mathcal{O}_2(z_2,\bar{z}_2)\cdots\mathcal{O}_n(z_n,\bar{z}_n)\rangle\,. \tag{4.13}$$

- Similar arguments as before may then be used to show that the spacetime dependence of two and three point function is completely fixed by conformal invariance.

  ▶▶ **Exercise 4.8:** *Rewrite* (3.49) *and* (3.50) *in terms for the 2d case with operators of given conformal weights. Note that holomorphicity brings further restrictions.*

  ▶▶ **Exercise 4.9:** *Rewrite the most general form of a four point function* (3.51) *for the 2d case.*

  NOTE: *The 2pt-func'n has a general form*

$$\langle \mathcal{O}_1(z,\bar{z})\mathcal{O}_2(w,\bar{w})\rangle = \frac{N_{12}}{(z-w)^{2h}(\bar{z}-\bar{w})^{2\bar{h}}}.$$

  For non-integer $2h$ and $2\bar{h}$ the "propagator" will have branch cuts. ***Think*** *about the features this issues may bring, e.g. like "parafermion" and general spin-statistics behavior and so on.*

4.3 Ward identities for 2d conformal invariance

- Ward identities associated with conformal invariance are those associated with Poincaré symmetries (3.41) and (3.43), plus those associated with scaling (3.44) and special conformal transformations.
- The first two, as well as the last two, are expressed in terms energy momentum tensor.
- In 2d these expressions take a simpler form. To analyze this, we start with

$$T_{zz} \equiv T(z,\bar{z})\,, \qquad T_{\bar{z}\bar{z}} \equiv \bar{T}(z,\bar{z}) = T(z,\bar{z})^* \tag{4.14}$$

  ▶▶ **Exercise 4.10:** *Show that the above are basically the reality conditions for energy momentum tensor.*

- *At classical level*, conformal invariance implies that

$$\begin{aligned} \text{E-M conservation:}\quad & \bar{\partial}T + \partial T_{\bar{z}z} = 0\,, \qquad \partial\bar{T} + \bar{\partial}T_{z\bar{z}} = 0\,,\\ \text{Scale invariance:}\quad & T_{z\bar{z}} + T_{\bar{z}z} = 0\,,\\ \text{Spin conservation:}\quad & T_{z\bar{z}} - T_{\bar{z}z} = 0\,. \end{aligned} \tag{4.15}$$

▶▶ **Exercise 4.11:** *Rewrite the above in Cartesian coordinate system. Write out the above conservation relations for Lorentzian signature.*

▶▶ **Exercise 4.12:** *What about the special conformal transformation invariance?*

- *At classical level*, the 2d conformal invariance hence imply that

$$\begin{aligned} &\bar{\partial}T = 0 \quad \Rightarrow T = T(z)\,, \qquad \partial\bar{T} = 0 \quad \Rightarrow \bar{T} = \bar{T}(\bar{z})\,, \\ &T_{z\bar{z}} = T_{\bar{z}z} = 0\,. \end{aligned} \tag{4.16}$$

- *At quantum level*, we need to examine the Ward identities. The associated four Ward identities are hence

$$\text{E-M conservation:} \quad \langle(\bar{\partial}T + \partial T_{\bar{z}z})\ \mathbf{X}\rangle = -\sum_{i=1}^{n} \delta^2(z - z_i)\ \partial_i\langle\mathbf{X}\rangle \tag{4.17}$$

$$\langle(\partial\bar{T} + \bar{\partial}T_{z\bar{z}})\ \mathbf{X}\rangle = -\sum_{i=1}^{n} \delta^2(z - z_i)\ \bar{\partial}_i\langle\mathbf{X}\rangle\,,$$

$$\text{Scale invariance:} \quad \langle(T_{z\bar{z}} + T_{\bar{z}z})\ \mathbf{X}\rangle = -\sum_{i=1}^{n} \delta^2(z - z_i)\ \Delta_i\langle\mathbf{X}\rangle\,, \tag{4.18}$$

$$\text{Spin conservation:} \quad \langle(T_{\bar{z}z} - T_{z\bar{z}})\ \mathbf{X}\rangle = -\sum_{i=1}^{n} \delta^2(z - z_i)\ s_i\langle\mathbf{X}\rangle\,, \tag{4.19}$$

where

$$\mathbf{X} \equiv \mathcal{O}_1(z_1, \bar{z}_1)\mathcal{O}_2(z_2, \bar{z}_2)\cdots\mathcal{O}_n(z_n, \bar{z}_n)$$

and Δ_i and s_i are scaling dimension and spin of the (quasi)primary operators $\mathcal{O}_i$.

▶▶ **Exercise 4.13:** *Recalling* (4.8)*, show that variation of* $\mathbf{X}$ *under conformal transformation is*

$$\delta_{\xi,\bar{\xi}}\mathbf{X} = -\sum_{i=1}^{n} (h_i\partial_i\xi\mathbf{X} + \xi\partial_i\mathbf{X}) + (\bar{h}_i\bar{\partial}_i\bar{\xi}\mathbf{X} + \xi\bar{\partial}_i\mathbf{X}) \tag{4.20}$$

where $\mathbf{X} = \mathcal{O}_1(w_1)\mathcal{O}_2(w_2)\cdots\mathcal{O}_n(w_n)$ *with* $\mathcal{O}_i$ *being local primary operators of weight* $(h_i, \bar{h}_i)$*.*

- Last two equations can also be written as

$$\langle T_{\bar{z}z}\ \mathbf{X}\rangle = -\sum_{i=1}^{n} \delta^2(z - z_i)\ h_i\langle\mathbf{X}\rangle\,, \qquad \langle T_{z\bar{z}}\ \mathbf{X}\rangle = -\sum_{i=1}^{n} \delta^2(z - z_i)\ \bar{h}_i\langle\mathbf{X}\rangle\,, \tag{4.21}$$

- Next we note that in 2d

$$\Box \ln|z - w|^2 = \partial\bar{\partial}\ln|z - w|^2 = 2\pi i\delta^2(z - w)\,, \tag{4.22}$$

and hence one can formally write

$$\delta^2(z-w) = \frac{1}{\pi}\partial_{\bar z}\frac{1}{z-w} = \frac{1}{\pi}\partial_z\frac{1}{\bar z - \bar w}$$

NOTE: *In 2d, one can always rewrite metric in a conformally flat form:* $ds^2 = F(z,\bar z)dzd\bar z$*. For this choice* $\Box = \frac{1}{F}\partial\bar\partial$*.*

- Using the above, Ward identities then reduce to

$$\partial_{\bar z}\left[\langle T\ \mathbf{X}\rangle - \sum_{i=1}^{n} \frac{1}{z-w_i}\partial_{w_i}\langle\mathbf{X}\rangle + \frac{h_i}{(z-w_i)^2}\langle\mathbf{X}\rangle\right] = 0\,, \tag{4.23}$$

$$\partial_{z}\left[\langle \bar T\ \mathbf{X}\rangle - \sum_{i=1}^{n} \frac{1}{\bar z-\bar w_i}\partial_{\bar w_i}\langle\mathbf{X}\rangle + \frac{\bar h_i}{(\bar z-\bar w_i)^2}\langle\mathbf{X}\rangle\right] = 0 \tag{4.24}$$

NOTE: *In the above we have absorbed a factor of* -2π *into the definition of* T *and* $\bar T$*.*

►► **Exercise 4.14:** *Work out the details and explicitly show the above.*

- One may hence arrive at

$$\langle T\ \mathbf{X}\rangle = \sum_{i=1}^{n} \frac{1}{z-w_i}\partial_{w_i}\langle\mathbf{X}\rangle + \frac{h_i}{(z-w_i)^2}\langle\mathbf{X}\rangle + regular\,, \tag{4.25}$$

$$\langle \bar T\ \mathbf{X}\rangle = \sum_{i=1}^{n} \frac{1}{\bar z-\bar w_i}\partial_{\bar w_i}\langle\mathbf{X}\rangle + \frac{\bar h_i}{(\bar z-\bar w_i)^2}\langle\mathbf{X}\rangle + regular\,. \tag{4.26}$$

- In particular if $\mathbf{X}$ is ANY given local primary operator $\mathcal{O}(w)$ of conformal weights $(h,\bar h)$:

$$\langle T\ \mathcal{O}(w,\bar w)\rangle = \frac{1}{z-w}\partial_w\langle\mathcal{O}(w,\bar w)\rangle + \frac{h}{(z-w)^2}\langle\mathcal{O}(w,\bar w)\rangle + regular\,, \tag{4.27}$$

$$\langle \bar T\ \mathcal{O}(w,\bar w)\rangle = \frac{1}{\bar z-\bar w}\partial_{\bar w}\langle\mathcal{O}(w,\bar w)\rangle + \frac{\bar h}{(\bar z-\bar w)^2}\langle\mathcal{O}(w,\bar w)\rangle + regular\,. \tag{4.28}$$

NOTE: *The above equations, the conformal symmetry Ward identities may be thought as the conditions for having a non-anomalous conformal symmetry, OR as the definition of primary operators.*

- Using (4.20) one can write the conformal Ward identities in a compact form:

$$\delta_{\xi,\bar\xi}\langle\mathbf{X}\rangle = -\frac{1}{2\pi i}\oint_C dz\xi(z)\langle T(z)\ \mathbf{X}\rangle + \frac{1}{2\pi i}\oint_C d\bar z\bar\xi(\bar z)\langle\bar T(\bar z)\ \mathbf{X}\rangle \tag{4.29}$$

where the contour C should include all w_i's, the insertions of $\mathcal{O}_i$.

NOTE: *The signs of the contour integrals are with the counter-clock-wise orientation for the contours.*

►► **Exercise 4.15:** *Simply the above for ξ in the $SL(2,C)$ part of the conformal algebra, i.e. for $\xi = a + bz + cz^2$, and for $\mathbf{X}$ being a product of TWO or THREE primary operators. Solve these equations to obtain the results of section 4.2.*

4.4 Operator Product Expansion (OPE)

- The above equations is usually denoted by the Operator Product Expansion **OPE**. For a general product of two operators in QFTs a relation of the form

$$\mathcal{O}_i(x)\mathcal{O}_j(y) = \sum_k C_{ijk}(x-y)\mathcal{O}_k(0)\,, \tag{4.30}$$

where the sum is over all possible local operators. The above operator-valued equation in any QFT is called OPE.

- In an QFT, the OPE coefficients usually are of the form $C_{ijk} \sim \frac{1}{|x-y|^{\Delta}}$.

►► **Exercise 4.16:** *For a unitary CFT, and assuming that $\mathcal{O}_i$ are primary local operators,*
I. *What is maximum value of Δ?*
II. *What is the minimum value of Δ?*
III. *Compute the x dependence of OPE coefficients C_{12i}.*

- Physically the main contribution to the OPE sum comes from positive Δ's. As the scaling dimension of the operator in the RHS increases, as Δ decreases and the contribution of the corresponding operator also decreases. That is, the "lightest" operators in the RHS have the largest contribution. This is the most *singular* terms in the OPE. We hence usually only focus on the largest Δ terms in the RHS and instead of equality use $\sim$, i.e.

$$\mathcal{O}_1(x_1)\mathcal{O}_2(x_2) \sim \frac{N_{12}}{|x_1-x_2|^{\Delta_1+\Delta_2}} + \frac{C_{12\Delta_0}}{|x_1-x_2|^{\Delta_1+\Delta_2-\Delta_0}}\mathcal{O}_{\Delta_0}(x_1) + \cdots \tag{4.31}$$

where Δ_0 is the lowest allowed dimension for unitary operators.

- OPE's are particularly useful in 2d CFTs, as we have separate left and right movers, the holomorhpic and anti-holomorphic parts.

- For example, the Ward identities (4.27) can be written as

$$T\ \mathcal{O}(w) \sim \frac{h}{(z-w)^2}\mathcal{O}(w) + \frac{1}{z-w}\partial_w\mathcal{O}(w)\,, \tag{4.32}$$

$$\bar{T}\ \mathcal{O}(w) \sim \frac{\bar{h}}{(\bar{z}-\bar{w})^2}\mathcal{O}(w) + \frac{1}{\bar{z}-\bar{w}}\partial_{\bar{w}}\mathcal{O}(w)\,. \tag{4.33}$$

4.4.1 OPE and operator commutators in 2d CFT

Operator ordering in radial quantization:

- In the radial quantization distance from the origin plays the role of time and hence the time-ordered product of operators (denoted by $\mathcal{R}$ is of the form:

$$\mathcal{R}(\mathcal{O}_1(z)\mathcal{O}_2(w)) = \begin{cases} \mathcal{O}_1(z)\mathcal{O}_2(w) & |w| > |z| \\ \\ \mathcal{O}_2(w)\mathcal{O}_1(z) & |z| > |w| \end{cases} \tag{4.34}$$

  and similarly for anti-holomorphic operators.

- we can extract coefficients of the Laurent expansion of operators by integrating them over appropriate powers of z or $\bar{z}$ (see (4.40)). Let us e.g. consider

$$\hat{\mathcal{O}}_1 = \oint dz\ \mathcal{O}_1(z)\,, \qquad \hat{\mathcal{O}}_2 = \oint dz\ \mathcal{O}_2(z)$$

  Then, one can show that

$$[\hat{\mathcal{O}}_1, \hat{\mathcal{O}}_2] = \oint_{Origin} dw \oint_w dz\ \mathcal{O}_1(z)\mathcal{O}_2(w)\,. \tag{4.35}$$

  ►► **Exercise 4.17:** *Complete the argument leading to the above. To this end, note that in second integral there are parts in which $|w| > |z|$ and parts which $|z| > |w|$. This integral may then be written in terms of the* difference *of two contour integrals where in one $|w| > |z|$ and in the other $|z| > |w|$.*

- Noting (4.35) the operator commutators can be worked out using OPE: the integrand can be replaced by its OPE expansion.

4.5 Virasoro generators and energy momentum tensor

- As we discussed the conformal group in any dimension, including 2d, has a geometrical origin. This may be used to construct conformal algebra as the Lie bracket of the diffeomorphism generating it. As we did so in section 2.

- Conformal group, on the other hand, may be viewed as symmetries of a given QFT, and one may hence use Noether theorem to read the corresponding conserved charges This was worked out in (3.33). This definition may be used to provide a representation of the conformal algebra through the fields (operators) in a QFT; a representation of conformal algebra over the field operators.

- One can work out the representation of 2d conformal algebra in terms of 2d CFT fields.

- For this case, one should a little bit "modify" the definition of the conserved charge in terms of integrals of Noether currents. This is to be done such that it takes care of holomorphic and anti-holomorphic parts separately. Explicitly, if we have a conserved current with holomorphic part $J(z)$ and anti-holomorphic part $\bar{J}(\bar{z})$, then

$$Q = \frac{1}{2\pi i}\oint dz\ J(z)\,, \qquad \bar{Q} = \frac{1}{2\pi i}\oint d\bar{z}\ \bar{J}(\bar{z})\,. \tag{4.36}$$

- For the 2d conformal group, the counterpart of (3.33) is hence

$$Q_\xi = \frac{1}{2\pi i}\oint dz\ T(z)\xi(z)\,, \qquad Q_{\bar{\xi}} = \frac{1}{2\pi i}\oint d\bar{z}\ \bar{T}(\bar{z})\bar{\xi}(\bar{z})\,, \tag{4.37}$$

where $\xi(z)$ is an arbitrary meromorphic function.

- Using the Laurent expansion of ξ, we obtain charges associated with each $\xi_n z^{(n+1)}$ term, L_n:

$$L_n = \frac{1}{2\pi i}\oint dz\ T(z)z^{(n+1)}\,, \qquad \bar{L}_n = \frac{1}{2\pi i}\oint d\bar{z}\ \bar{T}(\bar{z})\bar{z}^{(n+1)} \qquad n\in\mathbb{Z}. \tag{4.38}$$

The choice of z^{n+1} to be identified with L_n is for later convenience and to match with our previous notation for the Virasoro algebra.

▶▶ **Exercise 4.18:** *Invert the above relations to find:*

$$T(z) = \sum_{n=-\infty}^{+\infty} L_n z^{-n-2}\,, \qquad \bar{T}(\bar{z}) = \sum_{n=-\infty}^{+\infty} \bar{L}_n \bar{z}^{-n-2}\,. \tag{4.39}$$

▶▶ **Exercise 4.19:** *Use the Ward identity OPE* (4.32) *and the above definition* (4.39), *to find transformation of a primary operator* (4.9)*. This justifies, once again, that L_n's defined in* (4.38) *and those defined earlier in terms of conformal Killing diffeomorphism are indeed generators of the same algebra, the 2d conformal algebra, but defined on two different spaces.*

- One may use operator language one step further and expand any given primary operator $\mathcal{O}(z,\bar{z})$ of given weight $(h,\bar{h})$ as

$$\mathcal{O}(z,\bar{z}) = \sum_{n,m\in\mathbb{Z}} \mathcal{O}_{n,m} z^{-(n+h)}\bar{z}^{-(m+\bar{h})}\,. \tag{4.40}$$

▶▶ **Exercise 4.20:** *Using this notion show that* (4.9) *reduces to*

$$[L_n, \mathcal{O}_{m,p}] = [n(h-1)-m]\,\mathcal{O}_{n+m,p}\,, \quad [\bar{L}_n, \mathcal{O}_{m,p}] = \big[n(\bar{h}-1)-p\big]\,\mathcal{O}_{m,n+p}\,. \tag{4.41}$$

Alternative, directly use (4.35), (4.32) *and* (4.40), (4.38) *to compute the commutator* (4.41).

- Given that the generators of the Virasoro algebra L_n's are Laurent expansion coefficients of the energy momentum tensor T, one can work out the OPE of two T's:

$$
\begin{aligned}
T(z)T(w) &\sim \frac{\frac{\mathbf{c}}{2}}{(z-w)^4} + \frac{2}{(z-w)^2}T(z) + \frac{1}{z-w}\partial T(z)\,, \\
\bar{T}(\bar{z})\bar{T}(\bar{w}) &\sim \frac{\frac{\bar{\mathbf{c}}}{2}}{(\bar{z}-\bar{w})^4} + \frac{2}{(\bar{z}-\bar{w})^2}\bar{T}(\bar{z}) + \frac{1}{\bar{z}-\bar{w}}\bar{\partial}\bar{T}(\bar{z})\,.
\end{aligned}
\tag{4.42}
$$

If it were not for the first term, T would have been a primary operator of dimension two. That is, T is not a primary operator.

►► **Exercise 4.21:** *Recalling that T is related to generators of conformal algebra, argue that it is expected T to have dimension 2.*

►► **Exercise 4.22:** *Assuming that $\langle T(z)\rangle = 0$ show that*

$$
\begin{aligned}
\langle T(z)T(w)\rangle &= \frac{\frac{\mathbf{c}}{2}}{(z-w)^4}\,, \\
\langle T(z)T(w)T(u)\rangle &= \frac{\mathbf{c}}{(z-w)^2(z-u)^2(w-u)^2}
\end{aligned}
\tag{4.43}
$$

►► **Exercise 4.23:** *Recalling (4.37), work out variation of T under infinitesimal conformal transformation generated by $\xi(z)$.*
Answer:

$$
\delta_\xi T(z) = -\frac{\mathbf{c}}{12}\partial_z^3\xi(z) - 2\partial\xi(z)T(z) - \xi(z)\partial_z T(z)\,. \tag{4.44}
$$

►► **Exercise 4.24:** *Use the above to show that for FINITE conformal transformation $z \to w = w(z)$,*

$$
T'(w) = \left(\frac{dw}{dz}\right)^{-2}\left[T(z) - \frac{\mathbf{c}}{12}\{w;z\}_{Schwarzian}\right]\,, \tag{4.45}
$$

where $\{w;z\}_{Schwarzian}$ is the Schwarzian derivative of w w.r.t. z:

$$
\{w;z\}_{Schwarzian} = \frac{w'''}{w'} - \frac{3}{2}\left(\frac{w''}{w'}\right)^2\,, \tag{4.46}
$$

where prime denotes derivative w.r.t. z.

►► **Exercise 4.25:** *Show that for $SL(2,C)$ part of 2d conformal transformations,*

$$
z \to w = \frac{az+b}{cz+d}\,, \quad ad - bc = 1\,,
$$

the Schwarzian derivative vanishes.

- In view of the above exercise we conclude that *the energy momentum tensor, while not a conformal primary, it is quasi-primary.*

►► **Exercise 4.26:** *Find $w = w(z)$ such that $\left(\frac{dw}{dz}\right)^{-2}\{w;z\}_{Schwarzian}$ is a constant.*

4.6 2d CFT example, free boson theory

- To gain more physical picture of the formal stuff we discussed so far let us consider an example of free bosonic theory. The theory is described by the action:

$$S = \frac{1}{4\pi}\int d^2z\ \partial\phi_a\bar{\partial}\phi_a\,, \qquad a = 1, 2, \cdots, N\,. \tag{4.47}$$

- The conformal dimension of the ϕ_a fields read from the action is $(0,0)$. Therefore, $\partial\phi_a$, $\bar{\partial}\phi_a$ are respectively primary fields of weight $(1,0)$ and $(0,1)$.

- E.o.M and mode expansion:

$$\Box\phi_a = 0 \quad \Rightarrow \quad \phi_a = \phi_a(z) + \bar{\phi}_a(\bar{z})\,, \tag{4.48}$$

where

$$\phi_a(z) = \phi_0^a + \frac{1}{2}p^a \ln z + \frac{1}{\sqrt{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^a z^n\,, \qquad \bar{\phi}_a(\bar{z}) = \bar{\phi}_0^a + \frac{1}{2}\bar{p}^a \ln\bar{z} + \frac{1}{\sqrt{2}}\sum_{n\neq 0}\frac{1}{n}\bar{\alpha}_n^a\bar{z}^n\,. \tag{4.49}$$

NOTE: *The above is the solution with PERIODIC boundary conditions. That is, under $z \to e^{2\pi i}z$ it goes to itself, except for the $\ln z$ term.*

- The surface term coming from variation of the action (4.47) is

$$S_{Surface} = \frac{1}{4\pi}\int d^2z\ \partial(\delta\phi_a\bar{\partial}\phi_a) + \bar{\partial}(\delta\phi_a\partial\phi_a). \tag{4.50}$$

This term may be set to zero either by periodic boundary conditions or with Neumann or Dirichlet boundary conditions. See the exercise at the end of subsection 4.6.1 for quantization of the theory with other boundary conditions.

- To perform canonical quantization we need to know the conjugate momenta,

$$\begin{gathered}\Pi^a(z,\bar{z}) = \pi^a(z) + \bar{\pi}^a(\bar{z})\,,\\ \pi^a(z) = \frac{p^a}{2z} + \frac{1}{\sqrt{2}}\sum_{n\neq 0}\alpha_n^a z^n\,, \qquad \bar{\pi}^a(\bar{z}) = \frac{\bar{p}^a}{2z} + \frac{1}{\sqrt{2}}\sum_{n\neq 0}\bar{\alpha}_n^a\bar{z}^n\,.\end{gathered} \tag{4.51}$$

NOTE: *As we see $\pi(z)$ and $\bar{\pi}(\bar{z})$ have the Laurent expansion and are meromorphic as expected from a primary field.*

- Imposing canonical quantization commutation relations, we obtain

$$\begin{aligned} [\phi_0^a, p_b] &= i\delta_b^a\,, \qquad [\alpha_n^a, \alpha_m^b] = n\delta^{ab}\delta_{m+n}\,,\\ [\bar{\phi}_0^a, \bar{p}_b] &= i\delta_b^a\,, \qquad [\bar{\alpha}_n^a, \bar{\alpha}_m^b] = n\delta^{ab}\delta_{m+n}\,,\\ [\phi_0^a, \bar{p}_b] = [\bar{\phi}_0^a, p_b] &= 0\,, \qquad [\alpha_n^a, \bar{\alpha}_m^b] = 0\,. \end{aligned} \tag{4.52}$$

Since the holomorphic and anti-holomorphic parts decouple hereafter we only consider the holomorphic part.

▶▶ **Exercise 4.27:** *Work out* (4.52).

▶▶ **Exercise 4.28:** *Show that the energy momentum tensor T and $\bar{T}$ are given by:*

$$T = - : \partial\phi_a \partial\phi_a : , \qquad \bar{T} = - : \bar{\partial}\phi_a \bar{\partial}\phi_a : \tag{4.53}$$

where : : denotes the normal ordering.
NOTE: *The normal ordering is needed because we should choose the vacuum state such that $\langle T \rangle = \langle \bar{T} \rangle = 0$.*

▶▶ **Exercise 4.29:** *Work out the Virasoro generators using the above expressions for T and $\bar{T}$ and* (4.38)*:*

$$\begin{aligned} L_0 &= \frac{1}{2} p^a p^a \delta_n + \sum_{n>0} \alpha^a_{-n} \alpha^a_n \,, \qquad \bar{L}_0 = \frac{1}{2} \bar{p}^a \bar{p}^a \delta_n + \sum_{n>0} \bar{\alpha}^a_{-n} \bar{\alpha}^a_n \,, \\ L_n &= \sum_{m \in \mathbb{Z}} \alpha^a_{n-m} \alpha^a_m : , \qquad \bar{L}_n = \sum_{m \in \mathbb{Z}} : \bar{\alpha}^a_{n-m} \bar{\alpha}^a_m : , \; n \neq 0 \,, \end{aligned} \tag{4.54}$$

Work out the normal ordering explicitly and find the "zero point energy".

▶▶ **Exercise 4.30:** *Show that*

$$\alpha^a_n |0\rangle = \bar{\alpha}^a_n |0\rangle = 0 \,, n > 0 \quad \Rightarrow \quad L_n |0\rangle = 0 \,, n \geq -1. \tag{4.55}$$

NOTE: $p_a|0\rangle$, $\bar{p}_a|0\rangle$ *is not necessarily zero.*

- One may then construct states/operators which are eigen-states of p_a and $\bar{p}_a$. These are called *VERTEX* operators:

$$V_k(z, \bar{z}) =: e^{i k_a \phi_a(z, \bar{z})} : \tag{4.56}$$

▶▶ **Exercise 4.31:** *Show that*

$$p_a(V_k(0)|0\rangle) = k_a(V_k(0)|0\rangle). \tag{4.57}$$

Note that in the above the vacuum state is defined such that $p_a|0\rangle = 0$.

▶▶ **Exercise 4.32:** *Show that the above vertex operator is a primary state with conformal weights (h, h), $h = k^2/2$.*

▶▶ **Exercise 4.33:** *Work out the following OPE relation*

$$V_k(z, \bar{z}) V_p(w, \bar{w}) \sim |z - w|^{2k \cdot p} V_{k+p}(w, \bar{w}) \,, \tag{4.58}$$

▶▶ **Exercise 4.34:** *Correlation function of n vertex operators. Show that*

$$\langle V_{\vec{k}_1}(z_1,\bar{z}_1)V_{\vec{k}_2}(z_2,\bar{z}_2)\cdots V_{\vec{k}_n}(z_n,\bar{z}_n)\rangle = \prod_{i<j} |z_i-z_j|^{2\vec{k}_i\cdot\vec{k}_j}\ \delta(\sum_i \vec{k}_i)\,, \tag{4.59}$$

where $\vec{k}_i$ are N-vectors .

▶▶ **Exercise 4.35:** *Work out the commutation relations of L_n and read the central charges:*

$$\mathbf{c} = N\,, \qquad \bar{\mathbf{c}} = N\,. \tag{4.60}$$

- As we see the central charge is equal to the number of fields.
- We can work in path integral quantization as well. To do so we need to start with

$$\langle\phi_a(z,\bar{z})\phi_b(w,\bar{w})\rangle = -\frac{1}{2}(\ln(z-w)+\ln(\bar{z}-\bar{w})) + const\,, \tag{4.61}$$

or equivalently,

$$\begin{aligned}\langle\partial\phi_a(z)\partial\phi_b(w)\rangle &= -\frac{1}{2}\frac{1}{(z-w)^2} \sim \partial\phi_a(z)\partial\phi_b(w)\,,\\ \langle\bar{\partial}\phi_a(\bar{z})\bar{\partial}\phi_b(\bar{w})\rangle &= -\frac{1}{2}\frac{1}{(\bar{z}-\bar{w})^2} \sim \bar{\partial}\phi_a(z)\bar{\partial}\phi_b(w)\,.\end{aligned} \tag{4.62}$$

In the above the second equalities have been written as an OPE.

▶▶ **Exercise 4.36:** *Work out the following OPEs:*

$$T(z)\partial\phi_a(w) \sim \frac{\partial\phi_a(z)}{(z-w)^2} \sim \frac{\partial\phi_a(w)}{(z-w)^2} + \frac{\partial^2\phi_a(w)}{z-w}\,. \tag{4.63}$$

i.e. $\partial\phi_a$ is a primary of weight $(1,0)$.

▶▶ **Exercise 4.37:** *Consider the operator $:F(\phi):$. What is the condition on F is we demand this operator to be a primary. What is the relation between its conformal weights $h_F,\ \bar{h}_F$?*

- Using the above and (4.53) one can work out $T(z)T(w)$ OPE:

$$\begin{aligned}T(z)T(w) =&: \partial\phi_a(z)\partial\phi_a(z) :: \partial_b\phi(w)\partial\phi_b(w) :\\ &= 2\langle\partial\phi_a(z)\partial\phi_b(w)\rangle\langle\partial\phi_a(z)\partial\phi_b(w)\rangle + 2\cdot 2\langle\partial\phi_a(z)\partial\phi_b(w)\rangle : \partial\phi_a(z)\partial\phi_b(w) :\\ &\sim \frac{N/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{1}{z-w}\partial_z T(w)\,.\end{aligned} \tag{4.64}$$

4.6.1 Physical meaning of the central charge

NOTE: *The central charge* $\mathbf{c}$ *has many different interpretations and roles in the CFT's. For example:*

- $\mathbf{c}$ is usually attributed to number of dynamical fields and/or d.o.f of the 2d CFT.
- Another interpretation for $\mathbf{c}$ comes from the above exercise: $L_0 = \mathcal{L}_0 - \frac{\mathbf{c}}{24}$. $\mathcal{L}_0 + \bar{\mathcal{L}}_0$ (which generator of Witt algebra) is the Hamiltonian of the 2d CFT in the radial quantization on the plane, while $L_0 + \bar{L}_0$ is Hamiltonian on the cylinder. The difference between the two $\frac{\mathbf{c}+\bar{\mathbf{c}}}{24}$, is then attributed to **Casimir** energy of the theory on cylinder. This Casimir energy appears due to non-trivial (periodic) boundary conditions on the cylinder. In this interpretation, it is clearly seen that central charge has a *quantum* nature.

  To see the above, let us move from quantization on the 2d plane with $(z, \bar{z})$ to a cylinder. Upon the coordinate transformation (3.2)

  $$z \to w = \frac{R}{2\pi} \ln z \,, \tag{4.65}$$

  the plane is mapped to a cylinder of radius R. Under the above (4.45) takes the form

  $$T(z)_{pl} \to T(w)_{cyl} = \left(\frac{2\pi}{R}\right)^2 \left[T(z)_{pl} z^2 - \frac{\mathbf{c}}{24}\right] \,,$$

  and therefore,

  $$\langle T(w)_{cyl} \rangle = -\left(\frac{2\pi}{R}\right)^2 \frac{\mathbf{c}}{24} \,, \tag{4.66}$$

  IF $\langle T(z)_{pl} \rangle = 0$.

  ▶▶ **Exercise 4.38:** *Use (4.38) to read the Virasoro generators associated with the* T_{cyl}*. In particular show that*

  $$L_n^{cyl} = L_n^{pl} - \left(\frac{2\pi}{R}\right)^2 \frac{\mathbf{c}}{24} \delta_n \,. \tag{4.67}$$

- The above dovetails with (**??**).
- The above result may also be reached in the operator formulation and canonical quantization method, once we used radial (cylindrical) quantization. There we need to impose periodic boundary conditions and use ζ-function regularization for summing over the zero point energies of the oscillators (associated with each mode). That is, taking the steps of computing the Casimir energy. This will lead to $-\mathbf{c}/12$ for the zero point energy.

  ▶▶ **Exercise 4.39:** *Take the steps of computing the Casimir energy for a free boson theory on cylinder outlined above explicitly and compute the zero-point energy.*

▶▶ **Exercise 4.40:** *Primary operators on the cylinder vs. on the plane: The mode expansion* (4.40) *was given for a primary operator of conformal dimension* $(h,\bar{h})$ *on the plane. Write the same operator on the cylinder of radius* $R = 2\pi$ cf. (4.65) *and show that*

$$\mathcal{O}(w,\bar{w})_{cyl.} = \sum_{n,m\in\mathbb{Z}} \mathcal{O}_{n,m} e^{-wn} e^{-\bar{w}m} . \tag{4.68}$$

That is, the conventions in (4.40) *is chosen such that on the cylinder* $\mathcal{O}$ *has periodic boundary conditions, i.e. under* $w \to w + 2\pi i$ *and* $\bar{w} \to \bar{w} - 2\pi i$ *it remains invariant.*

- As mentioned the mode expansions and the analysis above are all true for *periodic* boundary conditions on the fields, i.e. $\phi_a(e^{2\pi i}z) = \phi_a(z)$.

▶▶ **Exercise 4.41:** *Show that for* ***anti-periodic*** *boundary conditions:*

$$\phi_a(e^{2\pi i}z) = -\phi_a(z) \tag{4.69}$$

the boundary term still vanishes and hence (4.69) *is a valid boundary condition.*
I. *Work out the mode expansion for* ϕ_a *and perform the canonical quantization procedure.*
II. *Work out the energy momentum tensor and Virasoro generators* L_n *coefficients.*
III. *Compute the zero point energy and the central charge for this case.*

▶▶ **Exercise 4.42:** *Show that for 2d CFTs on upper half-plane with* ***Neumann*** *boundary conditions:*

$$(\partial + \bar{\partial})\phi_a\big|_{Im\ z=0} = 0 \tag{4.70}$$

the boundary term still vanishes and hence (4.70) *is a valid boundary condition.*
I. *Work out the mode expansion for* ϕ_a *and perform the canonical quantization procedure.*
II. *Work out the energy momentum tensor and Virasoro generators* L_n *coefficients.*
III. *Compute the zero point energy and the central charge for this case.*

▶▶ **Exercise 4.43:** *Show that for 2d CFTs on upper half-plane with* ***Dirichlet*** *boundary conditions:*

$$\phi_a\big|_{Im\ z=0} = 0 \tag{4.71}$$

the boundary term still vanishes and hence (4.71) *is a valid boundary condition.*
I. *Work out the mode expansion for* ϕ_a *and perform the canonical quantization procedure.*
II. *Work out the energy momentum tensor and Virasoro generators* L_n *coefficients.*
III. *Compute the zero point energy and the central charge for this case.*

▶▶ **Exercise 4.44:** ***Orbifold theory****: The free bosonic action* (4.47) *has a global* $O(N)$ *symmetry acting on its fields. Let us for simplicity consider* $N = 2$ *case. Given this symmetry one has a bigger freedom in choosing the boundary conditions, because now the boundary term should vanish when we sum over a indices and not for each individual field. Consider the following boundary conditions:*

$$\phi_a(e^{2\pi i}z) = R_{ab}(\theta)\phi_b(z)\,, \qquad R_{ab}(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \tag{4.72}$$

where θ is an arbitrary angle.
I. *Show that* (4.72) *is a valid boundary condition.*
II. *Show that if we want standard periodic fields to also be included in* (4.72) *we should take* $\theta = 2\pi/k\,, k \in \mathbb{Z}$.
III. *Work out the mode expansion for* ϕ_a *and perform the canonical quantization procedure.*
IV. *Work out the energy momentum tensor and Virasoro generators* L_n *coefficients.*
V. *Compute the zero point energy and the central charge for this case.*

4.6.2 Deformation of a given 2d CFT

- So far we have discussed an example of free theory. One may add interaction terms to the 2d CFT to obtain more interesting theories.

- This may be done noting that Lagrangian is a $(1,1)$ primary operator. That is, deforming the theory by ANY primary $(1,1)$ operator we obtain (at least at classical level) a new 2d CFT. Therefore, there is a one-to-one relation between the spectrum of ALL $(1,1)$ operators and 2d CFT deformations.

- We also usually demand that the Lagrangian do not involve more that two derivative terms, due to appearance of ghosts in higher derivative theories.

- For the free bosonic theory discussed here, the $(1,1)$ operators which DOES NOT involve field derivatives are of the form
$$\mathcal{O}[f] = \int d^N k \delta(k^2-2) f(k_a) e^{ik_a\phi_a}\,, \qquad \forall f(k_a)\,. \tag{4.73}$$

- Given a generic $(1,1)$ operator like $\mathcal{O}[f]$, the action of the deformed theory is
$$\mathcal{L} = \frac{1}{4\pi}\partial\phi_a\bar{\partial}\phi_a + \mathcal{O}[f]\,. \tag{4.74}$$

- Note that the free Lagrangian has a global $O(N)$ symmetry acting on real-valued ϕ_a fields. Recalling that $O(N)$ has only one invariant two tensor (η_{ab}), generically the interaction term explicitly breaks, the global $O(N)$ symmetry. If $f(k_a)$ is non-zero for a single given N-vector k_a, then this symmetry is broken to the subgroup of k_a which keeps this vector invariant, i.e. $O(N-1)$.

- We can have $(1,1)$ operators which involve first derivatives of ϕ_a:
$$\mathcal{O}_{ab} =: F(\phi)\partial\phi_a\bar{\partial}\phi_b :\,, \quad \forall a,b,\ \&\ \text{some function } F(\phi)\,. \tag{4.75}$$

  ▶▶ Exercise 4.45:
  I. *What is the condition on* $F(\phi)$ *for the* $\mathcal{O}_{ab}$ *to be* $(1,1)$*?*
  II. *Write out the most general form of the deformation with* (4.75) *operators.*

  NOTE: *Although there are* $(1,1)$ *operators in the* $F(\phi)\partial\bar{\partial}\phi_a$ *family, since they vanish on-shell, do not provide a new form of deformation.*

4.7 Free 2d fermions

After discussing the 2d bosons, we consider 2d fermions. We should first construct irreducible 2d spinor representations. A brief discussion on construction of fermions in generic dimensions has appeared in the appendix A. Here we review the 2d case.

- In $2d$ **Dirac spinors** are $2^1 = 2$ component complex-valued spinors. The corresponding γ matrices are

$$\gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \qquad \Psi = \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix}. \tag{4.76}$$

  ►► **Exercise 4.46:** *Compute the explicit form of* $\gamma^z = (\gamma^1 + i\gamma^2)/2\,, \gamma^{\bar{z}} = (\gamma^z)^\dagger = (\gamma^1 - i\gamma^2)/2\,.$

- In 2d fermions can hence be Majorana-Weyl. $\Gamma = \sigma^3$ and therefore 2d Weyl fermions is one-complex valued component; in the notation above eg. Left handed spinor is ψ^1 while Right handed fermion is ψ^2. Imposing the Majorana condition is then easy: just take Ψ to be real-valued.

- In summary, Weyl-Majorana 2d spinors are real-valued one-component fields.

- Dirac action:

$$S = \frac{1}{2\pi}\int d^2z\ \bar{\Psi}\gamma^i\partial_i\Psi = \frac{1}{2\pi}\int d^2z\ \psi^1\bar{\partial}\psi^1 + \psi^2\partial\psi^2\,. \tag{4.77}$$

  NOTE: *As discussed, 2d fermions can be Weyl-Majorana and we may choose to work with only ψ^1 OR ψ^2 fields. Here for illustration purposes we considered 2d Dirac fermions.*

  NOTE: *The scaling dimension of 2d fermion is 1/2.*

- E.o.M:

$$\bar{\partial}\psi^1 = 0\,, \qquad \partial\psi^2 = 0\,. \tag{4.78}$$

  That is, the e.o.m implies that the Left handed fermion is holomorphic while the Right handed is anti-holomorphic:

$$\psi^1 = \psi^1(z)\,, \qquad \psi^2 = \psi^2(\bar{z})\,. \tag{4.79}$$

- Boundary conditions: Upon variation of the action the surface terms are

$$S_{surface} = \frac{1}{2\pi}\int d^2z\ \partial(\psi^1\delta\psi^1) + \bar{\partial}(\psi^2\delta\psi^2)\,, \tag{4.80}$$

The above surface term vanishes if we impose *periodic* or *anti-periodic* boundary conditions:

$$\text{Periodic bc}: \qquad \psi(e^{2\pi i}z) = -\psi(z)\,, \tag{4.81}$$

$$\text{Anti-periodic bc}: \qquad \psi(e^{2\pi i}z) = +\psi(z)\,. \tag{4.82}$$

NOTE: *The above notion of periodic and anti-periodic is on the CYLINDER.*
▶▶ **Exercise 4.47:** *Noting that fermion $\psi(z)$ is a $(1/2, 0)$ primary field, show that on the cylinder (cf. (4.65)) $\psi(w)$ is indeed periodic/anti-periodic.*

NOTE: *The above bc's has been written for holomorphic fermion fields. Similar bc's can of course be imposed on anti-holomorphic parts, independently of the ones imposed on the holomorphic fermions.*
NOTE: *In the context of string theory periodic and anti-periodic bc's are respectively called Ramond and Neveu-Schwarz boundary conditions.*

- Mode Expansions:

$$\text{Periodic, “Ramond”}: \qquad \psi(z) = \sum_{n\in\mathbb{Z}} d_n z^{-(n+1/2)}\,, \tag{4.83}$$

$$\text{Anti-periodic, “Neveu-Schwarz”}: \qquad \psi(z) = \sum_{r\in\mathbb{Z}+1/2} b_r z^{-(r+1/2)}\,. \tag{4.84}$$

- Quantization: The momentum conjugate to ψ is itself, as is seen from the action. Imposing the equal time canonical *anti-commutation* relations we find:

$$\{b_r, b_s\} = \delta_{r+s}\,, \qquad \{d_m, d_n\} = \delta_{m+n}\,. \tag{4.85}$$

- Energy momentum tensor:

$$T = -\frac{1}{2} : \psi^1 \partial \psi^1 :\,, \qquad \bar{T} = -\frac{1}{2} : \psi^2 \bar{\partial} \psi^2 : \,. \tag{4.86}$$

▶▶ **Exercise 4.48:** *Show that the Virasoro generators are*

$$\textit{NS sector:} \qquad L_n = \frac{1}{2}\sum_{s\in\mathbb{Z}+1/2}(s+\frac{1}{2}) : b_{n-s}b_s :\,, \quad L_0 = \sum_{r>0, r\in\mathbb{Z}+1/2} r b_{-r} b_r\,, \tag{4.87}$$

$$\textit{Ramond sector:} \qquad L_n = \frac{1}{2}\sum_{m\in\mathbb{Z}}(m+\frac{1}{2}) : d_{n-m}d_m :\,, \quad L_0 = \sum_{m>0, m\in\mathbb{Z}} m d_{-m} d_m + \frac{1}{16}\,, \tag{4.88}$$

▶▶ **Exercise 4.49:** *Compute $[L_n, b_r]$ and $[L_n, d_m]$ commutators and confirm that the ψ field is indeed a $(1/2, 0)$ primary field.*

- The central charge.
▶▶ **Exercise 4.50:** *Given the above expression for the L_n generators, work out the central charge of the theory for the Weyl-Majorana free fermion is $1/2$.*

- **Path Integral treatment** of fermionic system. We start with the two-point function for fermions:

$$\langle \psi^1(z)\psi^1(w)\rangle = \frac{1}{z-w}\,, \qquad \langle \psi^2(\bar z)\psi^2(\bar w)\rangle = \frac{1}{\bar z-\bar w}\,. \tag{4.89}$$

▶▶ **Exercise 4.51:** *Given the above show that*

$$T(z)\psi(w) \sim \frac{1/2}{(z-w)^2}\psi(w) + \frac{1}{z-w}\partial_w\psi(w)\,, \tag{4.90}$$

and also

$$T(z)T(w) \sim \frac{1/4}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}\,. \tag{4.91}$$

▶▶ **Exercise 4.52:** *The above fields* (4.83) *and* (4.84) *are written on the plane.*
I. *Write them on cylinder.*
II. *Write the two point function* (4.89) *for cylindrical fields.*
III. *Compute the energy momentum tensor for the theory on cylinder, T_{cyl}.*
IV. *Compute $\langle T\rangle$ and show that*

$$\langle T_{cyl}\rangle = \begin{cases} -\left(\frac{2\pi}{R}\right)^2\frac{1}{48} & \text{NS sector} \\ +\left(\frac{2\pi}{R}\right)^2\frac{1}{24} & \text{Ramond sector} \end{cases} \tag{4.92}$$

▶▶ **Exercise 4.53:** *Classify all the* $(1,1)$ *operators one can build out of fermion fields ψ^1 and ψ^2.*
Write all the $(1,1)$ *fields made out of a primary scalar field ϕ and fermionic fields ψ.*

- We discussed two examples of simple 2d CFTs to gain some intuition about the structure of 2d CFTs and their central charge, and classified all their perturbative deformations which preserve conformal symmetry at classical level.

- It yet remains to be checked if all such deformations which are parameterized by $(1,1)$ primary operators also keep conformal symmetry at quantum level.

- We computed the perturbative central charge, while it can in principle receive corrections from the deformations added to the free energy momentum tensor.

4.8 More on Unitary Rep's of Virasoro algebra

- There are different ways to classify 2d CFTs, one way is based on the value of their central charge and their physical Hilbert space, i.e. the spectrum of all primary operators. In this section we will try to see how much information we can extract from "algebraic" restrictions like Unitarity of representations of Virasoro algebra of a given central charge **c**, without delving into the details of the theory, or specifying the explicit form of the energy momentum tensor T.

- As we discussed earlier, unitarity (positivity of norm) of descendent of the vacuum $||L_n|0\rangle|| \geq 0$ for all $n \geq 2$ implies $\mathbf{c} \geq 0$.
- Positivity of norm for "level one" descendent of a primary state $|h\rangle$, i.e. $||L_{-1}|h\rangle|| \geq 0$ implies $h \geq 0$.
- As we discussed earlier in section 3, higher level descendents, which in the $d > 2$ cases can only be constructed by acting higher powers of P_μ on the primary, for level two leads to an extra condition on scalar reps and at higher levels leads to no extra condition. So the level two analysis in $d > 2$ exhausted all the unitarity conditions on the representations.
- In $d = 2$ case, however, there are many more ways to construct descendents of a given primary state: we have the option of acting by any combination of L_{-n}'s; that is how we construct states in the Verma module of a given primary. **NOTE:** *Compared to the $d > 2$ cases, that is the action of $L^m_{-1}\bar{L}^n_{-1}$ which creates level $l = m+n$ descendents corresponding to P^l_μ descendent.*
- To start, let us consider the unitarity of *level* K states in the Verma module of vacuum:

$$|\{k_i\}\rangle \equiv \prod_{i=1}^{K} L_{-k_i}|0\rangle\,, \qquad k_i \geq k_j \ \ if \ \ i > j\,, \ \sum k_i = K\,. \tag{4.93}$$

►► **Exercise 4.54:** *Using scale invariance show that* $\langle\{k_i\}|\{p_j\}\rangle \propto \delta(\sum k_i - \sum p_j)$.

- To show unitarity of all states in the vacuum Verma module one should hence show that *for all states in level* K the eigenvalues of the matrix

$$M^{vac}_K \equiv \langle\{k_i\}|\{p_j\}\rangle$$

are non-negative.
►► **Exercise 4.55:** *Show that M^{vac}_K is a $P_K \times P_K$ matrix where P_K is number of partitions of integer K into non-negative integers.*
►► **Exercise 4.56:** *Show that all elements in M^{vac}_K with one or more of k_i's equal to one leads to a zero eigenvalue in M^{vac}_K. Using this, find number of zero eigenvalues of M^{vac}_K.*

- One can show that all eigenvalues of M^{vac}_K are non-negative if $\mathbf{c} \geq 0$. Therefore, from the vacuum Verma module we find no extra condition on unitarity.
- Next, let us consider the Verma module of a given primary $|h\rangle$, $V(h;\mathbf{c})$:

$$|h;\{k_i\}\rangle \equiv \prod_{i=1}^{K} L_{-k_i}|h\rangle\,, \tag{4.94}$$

►► **Exercise 4.57:** *Using scale invariance show that*

$$\langle h;\{k_i\}|h;\{p_j\}\rangle \propto \delta(\sum k_i - \sum p_j).$$

- We again focus on

$$M_K^h \equiv \langle h; \{k_i\} | h; \{p_j\}\rangle\,, \qquad \sum k_i = K.$$

  – At level one, $K = 1$, we just get $h \geq 0$.

  – at level two, $K = 2$, the matrix M is 2×2:

$$M_{11} = \langle h|L_1^2 L_{-1}^2|h\rangle = 4h(2h+1)\,,\ M_{12} = \langle h|L_1^2 L_{-2}|h\rangle = 6h\,,\ M_{11} = \langle h|L_2 L_{-2}|h\rangle = 4h + \frac{\mathbf{c}}{2}\,. \tag{4.95}$$

  To have positive eigenvalues M should have positive trace and determinant. Trace is already positive. The determinant is

$$\det M_2^h = 32(h - h_{1,1})(h - h_{1,2})(h - h_{2,1}) = 32h\left[(h - \frac{1}{4})^2 + \frac{1}{16}(\mathbf{c} - 1)(h + 2)\right]\,, \tag{4.96}$$

  where

$$h_{1,1} = 0\,, \quad h_{1,2} = \frac{1}{16}\left[5 - \mathbf{c} + \sqrt{(1-\mathbf{c})(25-\mathbf{c})}\right]\,, \quad h_{2,1} = \frac{1}{16}\left[5 - \mathbf{c} - \sqrt{(1-\mathbf{c})(25-\mathbf{c})}\right]\,. \tag{4.97}$$

  As we see from the second equality in (4.96), the determinant is positive for any h IFF $\mathbf{c} \geq 1$.

  – Higher level, $K > 2$: One can show that

$$\det M_K^h = \mathcal{N}_K \prod_{\substack{p,q \geq 1 \\ pq \leq K}} (h - h_{p,q}(\mathbf{c}))^{P_{K-pq}}\,, \tag{4.98}$$

  where $\mathcal{N}_K$ is ia positive number, P_m is number of partitions of a given integer m into non-negative integers and

$$h_{p,q}(\mathbf{c}) = \frac{1}{24}(\mathbf{c} - 1) + \frac{1}{4}\left(pA_+ + qA_-\right)^2\,, \qquad A_\pm = \frac{1}{24}\left(\sqrt{1-\mathbf{c}} \pm \sqrt{25-\mathbf{c}}\right)\,. \tag{4.99}$$

  The above can also be written as

$$h_{p,q}(r) = \frac{(p(r+1) + qr)^2 - 1}{4r(r+1)}\,, \qquad \mathbf{c} = 1 - \frac{6}{r(r+1)}\,. \tag{4.100}$$

  NOTE: $\det M_K^h$ *is called* ***Kac determinant****.*

  – Kac determinant vanished if $h = h_{p,q}(\mathbf{c})$.

  ►► Exercise 4.58:
  I. *Compute* M_K^h *for large* h*, i.e. for* $h \gg K \sim 1$*.*
  II. *Show that for large* h *the largest element in* M_K^h *comes form* $L_{-1}^K|h\rangle$*.*
  III. *Diagonalize* M_K^h *for large* h*.*

- Given the Kac determinant one can now study the unitarity condition which is positivity of the determinant.

- Since $h > 0$,

$$h_{p,q}(\mathbf{c}) \leq 0 \quad \forall p, q \geq 1 \quad \Rightarrow \quad \det\ M_K^h \geq 0\,. \tag{4.101}$$

4.8.1 Unitary Rep's for $\mathbf{c} \geq 1$ case

We show below that the positive $h_{p,q}$ condition (4.101) holds for $\mathbf{c} \geq 1$ cases.

- If $1 \leq \mathbf{c} \leq 25$ then
$$A_{\pm} = \pm a + ib \qquad \Rightarrow \qquad h_{p,q}(\mathbf{c}) = h_{q,p}(\mathbf{c})^* \tag{4.102}$$
Next, recall that the terms in the Kac determinant are of the form $\left[(h - h_{p,q})(h - h_{q,p})\right]^P$ which are hence positive.

- For $\mathbf{c} > 25$ cases
$$A_{\pm} = i(a \pm b)\,, \qquad a^2 = \frac{\mathbf{c}-1}{24}\,,\; b^2 = \frac{\mathbf{c}-25}{24} = a^2 - 1\,. \tag{4.103}$$
and hence
$$h_{p,q}(\mathbf{c}) = a^2 - \frac{1}{4}\left[p(a+b) + q(a-b)\right]^2 \leq a^2 - \frac{1}{4}(a+b+a-b)^2 \leq 0\,. \tag{4.104}$$
where we used $p, q \geq 1$.

  So we proved the desired.

►► **Exercise 4.59:** *Show that for $\mathbf{c} = 1$ case, for $K = 2$ case we have zero norm state IF $h = 1/4$.*

►► **Exercise 4.60:** *Find the zero norm state of level two $K = 2$ for $\mathbf{c} < 1$ model. Find zero norm state(s) of level $K = 3$ of $\mathbf{c} < 1$ models.*

►► **Exercise 4.61:** *Show that for $\mathbf{c} > 1$ case we do not have zero norm states in the Verma module of primary $h > 0$. Show this for $K = 2, 3$ cases first and then generalize to higher K.*

4.8.2 Unitary Rep's for $\mathbf{c} < 1$ case

- In case case it is more convenient to use (4.100) parametrization in terms of r.

- In this parametrization $0 < \mathbf{c} < 1$ yields $r > 2$ or $r < -3$. Since
$$h_{p,q}(r) = \frac{\left[(r+1)p - rq\right]^2 - 1}{4r(r+1)} = h_{q,p}(-(r+1))\,, \tag{4.105}$$
and in the Kac determinant we cover all p, q cases, we can safely restrict ourselves to $r > 2$ region only.

- To avoid non-unitarity the only way is to make $h_{p,q}(r)$ to vanish at some point and cut the region where $h_{p,q}(r)$ becomes positive.

- Demanding $h_{p,q}(r)$ to have zeros for some $p, q \geq 1$ tells us that r should be **an integer**.

- Moreover, to remain in the negative $h_{p,q}(r)$ region, $1 \leq p < r$ and $1 \leq q < p$.
- It happens that the above conditions are necessary and sufficient for unitraity of $\mathbf{c} < 1$ models.
- In the $\mathbf{c} < 1$ theories, therefore, the scaling dimension of primaries are bound to be larger than $h_{p,q}(r)$ for any given $1 \leq p < r$, $1 \leq q < p$.

4.8.3 Minimal models

- There is a complete classification of $\mathbf{c} \leq 1$ 2d CFT's. A interesting class of $\mathbf{c} \leq 1$ models are **minimal models**. These are models with *finite number of primary states.*
- The unitarity requries that the weight of these primaries are $h \geq h_{p,q}$. So the lowest weight primaries are given by $h = h_{p,q}(r)$. The condition of having FINITE number of primaries then requires that

$$h_{p,q} = h_{p+m,q+n} \ , \qquad \text{for some integer } m, \ n\,. \tag{4.106}$$

The above holds IFF

$$\mathbf{c} = 1 - 6\frac{(m-n)^2}{mn}\,, \qquad h_{p,q} = \frac{(pm-qn)^2 - (m-n)^2}{4mn}\,, \tag{4.107}$$

where m, n are coprime integers larger or equal to two and $1 \leq p \leq n-1, \ 1 \leq q \leq m-1$.

▶▶ **Exercise 4.62:** *Determine m, n in terms of our earlier parametrization r. What does $r > 2$ correspond to in terms of m, n?*

NOTE: *Minimal models are hence specified by two coprime integers m, n and usually denoted by $\mathcal{M}(m, n)$.*

▶▶ **Exercise 4.63:** *Using* (4.106) *show that number of primary states in the minimal model given by (m, n) is $(m-1)(n-1)/2$.*

- Some of the most interesting stat.mech. or cond.mat. models around their critical point are examples of minimal models. These include, 2d Ising, 2d Potts model, $O(N)$ model, Landau Ginzburg model (i.e a scalar theory with $\phi^{2(r-1)}$ potential).

4.9 More on structure of OPE and, 3 and 4 point functions

- We discussed that OPE in local QFTs is a useful tool to analyze and compute correlators of the theory.
- In $d > 2$ CFTs conformal invariance can be used to restrict further the form of OPE coefficients.

- In general the OPE of two *primary* operators in 2d CFTs, as in higher dimensional examples, can be written as

$$\mathcal{O}_i(z,\bar{z})\mathcal{O}_j(0) = \sum C_{ijk}\mathcal{O}_k(0) z^{h_k-h_i-h_j}\bar{z}^{\bar{h}_k-\bar{h}_i-\bar{h}_j}, .$$

The notable point in the above OPE is that the operators $\mathcal{O}_k$ appearing in the RHS are not necessarily primary, while they are in general *quasi-primary*, i.e. they have definite conformal weight but can be a primary or descendent of a primary.

- On the other hand, we know the space-time dependence of three point function of **primary** operators $\mathcal{O}_1, \mathcal{O}_2,\ co_3$ is fixed by conformal symmetry:

$$\langle \mathcal{O}_1(z_1,\bar{z}_1)\mathcal{O}_2(z_2,\bar{z}_2)\mathcal{O}_3(z_3\,\bar{z}_3)\rangle = \frac{C^p_{123}}{\prod_{i<j,\ k\neq i,j}(z_i-z_j)^{h_i+h_j-h_k}\cdot(\bar{z}_i-\bar{z}_j)^{\bar{h}_i+\bar{h}_j-\bar{h}_k}} \tag{4.108}$$

where superscript p on C^p_{123} is to emphasize that this is for primary operators.

- The question is now how to read the OPE coefficients from the three point function of primaries.

4.9.1 OPE coefficients from three point function of primaries

- It turns out that one can deduce the OPE coefficients of descendents appearing in the RHS in terms of that of the corresponding primary. To see this let us rewrite the above OPE as

$$\mathcal{O}_i(z,\bar{z})\mathcal{O}_j(0) = \sum_{primary}\sum_{\{k_i\}}\sum_{\{\bar{k}_i\}} C^{\{k_i\},\{\bar{k}_i\}}_{ijp}\prod_{i=1,\bar{i}=1}^{K,\bar{K}} L_{-k_i}\bar{L}_{-\bar{k}_i}\mathcal{O}_p(0)\ z^{h_p+K-h_i-h_j}\bar{z}^{\bar{h}_p+\bar{K}-\bar{h}_i-\bar{h}_j} \tag{4.109}$$

where now $\mathcal{O}_p$ only involve primary operators and the whole Verma module above them.

- One may decompose the OPE coefficients in terms of primary and descendent parts:

$$C^{\{k_i\},\{\bar{k}_i\}}_{ijp} \equiv C_{ijp}\ \mathcal{N}^{p\{k_i\}}_{ij}\ \bar{\mathcal{N}}^{p\{\bar{k}_i\}}_{ij} \tag{4.110}$$

where we have chosen the normalization such that $\mathcal{N}^{p\{0\}}_{ij} = \bar{\mathcal{N}}^{p\{\bar{0}\}}_{ij} = 1$.

- One may then define the operator

$$\varphi^p_{ij}(z) \equiv \sum_{\{k_i\}}\ \mathcal{N}^{p\{k_i\}}_{ij} z^K\prod_{i=1}^{K} L_{-k_i} \tag{4.111}$$

and similarly for the anti-holomorphic part $\bar{\varphi}^p_{ij}(\bar{z})$.

- Using the above the OPE now takes the form:

$$\mathcal{O}_i(z,\bar{z})\mathcal{O}_j(0) = \sum_{primary} C_{ijp} z^{h_p-h_i-h_j}\ \bar{z}^{\bar{h}_p-\bar{h}_i-\bar{h}_j}\ \varphi^p_{ij}(z)\bar{\varphi}^p_{ij}(\bar{z})\ \mathcal{O}_p(0)\,. \tag{4.112}$$

So we have succeeded in the first step of reducing the OPE to only primary sector. The Verma module effects are now "hidden" in the operator-valued φ coefficients. **NOTE:** *The above OPE may also be written as*

$$\mathcal{O}_i(z,\bar{z})|\mathcal{O}_j\rangle = \sum_{primary} C_{ijp} z^{h_p-h_i-h_j}\ \bar{z}^{\bar{h}_p-\bar{h}_i-\bar{h}_j}\ \varphi^p_{ij}(z)\bar{\varphi}^p_{ij}(\bar{z})|\mathcal{O}_p\rangle\,. \tag{4.113}$$

▶▶ **Exercise 4.64:** *Noting that the three-point function of generic* ***primary*** *operators, and using the fact that one of the points can always be set to origin and the other taken to infinity, we have*

$$\begin{aligned} limit_{w,\bar{w}\to\infty}(w^{2h_i}\bar{w}^{2\bar{h}_i}\langle\mathcal{O}_i(w,\bar{w})\mathcal{O}_j(z,\bar{z})\mathcal{O}_k(0)\rangle) &= \langle\mathcal{O}_i|\mathcal{O}_j(z,\bar{z})|\mathcal{O}_k\rangle \\ &= C_{ijk} z^{h_i+h_k-h_j}\ \bar{z}^{\bar{h}_i+\bar{h}_k-\bar{h}_j}\,, \end{aligned} \tag{4.114}$$

show that the OPE coefficients C_{ijp} can be read from three point function of appropriate primaries. **NOTE:** *One can still use the remaining conformal transformation freedom to set $z=\bar{z}=1$ in* (4.114)*.*

- **Computing $\varphi^p_{ij}(z)$.** To this end, we need $\mathcal{N}^{p\{k_i\}}_{ij}$ coefficients. The latter are **c**-number coefficients which only depend on h_i, h_j and central charge **c** as well as the set of $\{k_i\}$.

▶▶ **Exercise 4.65:** *To compute $\varphi^p_{ij}(z)$ use the OPE* (4.113) *and act on both sides by Virasoro generators L_n. Recalling that $\mathcal{O}_i$ and $\mathcal{O}_j$ are primary we know the LHS. Using this find differential equations governing $\varphi^p_{ij}(z)$ and from there solve for $\mathcal{N}^{p\{k_i\}}_{ij}$ coefficients.*

4.9.2 4 point functions, conformal blocks

- In previous subsection we argued how the information of three point function of primary operators can be used to read the OPE coefficients of two primary operators.
- On the other hand, having the OPE one can reduce computation of four point function of primary operators to computation of two point functions.
- In this subsection we combine these information to compute four point functions. Consider a generic four point function of primaries

$$G^p_{1234}(z_i) = \langle\mathcal{O}_1(z_1,\bar{z}_1)\mathcal{O}_2(z_2,\bar{z}_2)\mathcal{O}_3(z_3,\bar{z}_3)\mathcal{O}_4(z_4,\bar{z}_4)\rangle\,,$$

and choose the $SL(2,C)$ freedom to map $z_1\to\infty$, $z_4=0$, $z_3=1$ and call $z_2=z$, and define

$$\mathcal{G}^p_{1234}(z,\bar{z}) \equiv \langle\mathcal{O}_1|\mathcal{O}_2(z,\bar{z})\mathcal{O}_3(1,1)|\mathcal{O}_4\rangle = limit_{z_1,\bar{z}_1\to\infty}\left(z_1^{2h_1}\bar{z}_1^{2\bar{h}_1}G^p_{1234}(z_1,z,1,0)\right) \tag{4.115}$$

▶▶ **Exercise 4.66:** *What is the relation between the two conformal cross-ratios and the coordinate z?*

The goal is hence computing $\mathcal{G}^p_{1234}$. **NOTE:** *Order of indices* 1234 *in* $\mathcal{G}$ *is important.*

- We can now use the OPE (4.113) twice for $\mathcal{O}_3(1,1)|\mathcal{O}_4\rangle$ and for $(\mathcal{O}^\dagger_2(z,\bar{z})|\mathcal{O}_1\rangle)^\dagger$ to compute $\mathcal{G}^p_{1234}(z,\bar{z})$. Explicitly, we have

$$\begin{aligned}\mathcal{G}^p_{1234}(z,\bar{z}) &= \sum_p \sum_{p'} C^*_{12p} C_{34p'}\ z^{h_p-h_1-h_2}\ \bar{z}^{\bar{h}_p-\bar{h}_1-\bar{h}_2}\ \langle \mathcal{O}_p|\varphi^p_{12}(z)^\dagger \bar{\varphi}^p_{12}(\bar{z})^\dagger \cdot \varphi^{p'}_{34}(1)\bar{\varphi}^{p'}_{34}(1)|\mathcal{O}_{p'}\rangle \\ &= \sum_p C^*_{12p} C_{34p}\ z^{h_p-h_1-h_2}\ \bar{z}^{\bar{h}_p-\bar{h}_1-\bar{h}_2}\ \langle \mathcal{O}_p|\chi^p_{1234}(z)\cdot \bar{\chi}^p_{1234}(\bar{z})|\mathcal{O}_p\rangle\,,\end{aligned} \tag{4.116}$$

where in the second line we used the Verma module orthogonality relation (4.12) and

$$\chi^p_{1234}(z) = \varphi^p_{12}(z)^\dagger \varphi^p_{34}(1)\,, \qquad \bar{\chi}^p_{1234}(\bar{z}) = \bar{\varphi}^p_{12}(\bar{z})^\dagger \bar{\varphi}^p_{34}(1)\,. \tag{4.117}$$

In the above we used the fact that Left and Right mover Virasoro generators commute.
NOTE: *In* (4.116) *we only remain with a sum over primaries, the information about Verma modules are hidden inside* χ_{1234} *coefficients.*
NOTE: *The holomorphic and anti-holomorphic parts of the amplitude are now decoupled.*

- Next, we use (4.111) to explicitly write χ_{1234} in terms of Virasoro generators

$$\chi^p_{1234}(z) = \sum_{\{k_i\}} \sum_{\{k'_i\}} (\mathcal{N}^{p\{k_i\}}_{12})^*\ \mathcal{N}^{p\{k'_i\}}_{34}\ z^K \prod_{i=1}^{K} L_{+k_i} \prod_{i=1}^{K'} L_{-k'_i} \tag{4.118}$$

As we see the operator part of these coefficients depend only on Virasoro generators and their numeric coefficients are only functions of the conformal weight of $\mathcal{O}_i$ operators and the central charge.

- It is customary to use the OPE expansion (4.113) only for $\langle \mathcal{O}_1|\mathcal{O}_2(z,\bar{z})$ (i.e. for the operator defined at z and keep $\mathcal{O}_3(1)|\mathcal{O}_4\rangle$ intact. This leads to

$$\mathcal{G}^p_{1234}(z,\bar{z}) = \sum_p C^*_{12p} C_{34p}\ z^{h_p-h_1-h_2}\ \bar{z}^{\bar{h}_p-\bar{h}_1-\bar{h}_2}\ \cdot \mathcal{A}^p_{12\to34}(z)\bar{\mathcal{A}}^p_{12\to34}(\bar{z}) \tag{4.119}$$

where

$$\mathcal{A}^p_{12\to34}(z) = \sum_{\{k_i\}} (\mathcal{N}^{p\{k_i\}}_{12})^* z^K \frac{\langle \mathcal{O}_p|\prod_{i=1}^K L_{+k_i}\mathcal{O}_3(1)|\mathcal{O}_4\rangle}{\langle \mathcal{O}_p|\mathcal{O}_3(1)|\mathcal{O}_4\rangle_{holo}} \tag{4.120}$$

where $\langle \mathcal{O}_p|\mathcal{O}_3(1)|\mathcal{O}_4\rangle_{holo} = \sqrt{C_{34p}}$. Similar expression also hold for anti-holomorphic part $\bar{\mathcal{A}}^p_{12\to34}(\bar{z})$.
NOTE: *As we note the holomorphic and anti-holomorphic parts are again decoupled.*
NOTE: *The coefficients* $\mathcal{A}^p_{12\to34}(z)$ *are called* ***conformal blocks***.

- The coefficients of Laurent expansion for the conformal blocks, $\mathcal{A}^{p,K}_{12\to 34}$

$$\mathcal{A}^{p}_{12\to 34}(z) = \sum_{K=0}^{\infty} z^K \mathcal{A}^{p,K}_{12\to 34},$$

can be computed. Here we list the first three:

$$\begin{aligned}
\mathcal{A}^{p,0}_{12\to 34} &= 1\,,\\
\mathcal{A}^{p,1}_{12\to 34} &= \frac{(h_p+h_2-h_1)(h_p+h_3-h_4)}{2h_p}\,,\\
\mathcal{A}^{p,2}_{12\to 34} &= \frac{(h_p+h_2-h_1)(h_p+h_3-h_4)(h_p+h_2-h_1+1)(h_p+h_3-h_4+1)}{4h_p(2h_p+1)}+\\
&+\frac{1}{8}\left[\left(h_1+h_2+\frac{h_p(h_p-1)-3(h_1-h_2)^2}{2h_p+1}\right)\left(h_3+h_4+\frac{h_p(h_p-1)-3(h_3-h_4)^2}{2h_p+1}\right)\right]^2\\
&\times\left(c+\frac{2h_p(8h_p-5)}{2h_p+1}\right)^{-1}\,. \qquad (4.121)
\end{aligned}$$

NOTE: *As we see the coefficients* $\mathcal{A}^{p,K}_{12\to 34}$ *show the* $12 \leftrightarrow 34$ *"exchange symmetry".*

►► **Exercise 4.67:** *One could have used to map* $z_2 = 1$ *and* $z_3 = z$. *Repeat the above with this choice and find the corresponding conformal blocks and in this way PROVE the exchange symmetry mentioned above.*

4.10 2d CFTs on torus and modular invariance

- 2d CFTs, as QFTs in any other dimension, can be defined on different manifolds.
- However, if the conformal symmetry is not anomalous, for a CFT this manifold may be defined up to conformal transformations.
- By very definition conformal transformations are only based on differential geometric notions like metric and diffeomorphism. However, to define a geometry we also need information about the global topological structure.
- In this part we study 2d CFT on a non-trivial topology and in particular two torus T^2. This is relevant for non-zero temperature 2d CFTs. Before studying this problem, we make a detour to classification of 2d surfaces.

4.10.1 Classification of 2d surfaces

- Euclidean 2d surfaces may be classified by their *metric, topology and orientablility.*

- **Metric:** Local properties of manifolds are specified by their metric. In 2d, upon the choice of coordinates, metric can always be *locally* brought to conformally flat form:
$$ds^2 = e^{2\phi}(dx^2 + dy^2) = e^{2\phi} dz d\bar{z} \,. \tag{4.122}$$
▸▸ Exercise 4.68: *Write down the metric of a round two-sphere in the above form.*
In 2d Riemann curvature is specified by a single number, and has essentially the same information as the Ricci scalar. In the conformally flat metric that is
$$\mathcal{R} = 2e^{-2\phi}\partial\bar{\partial}\phi \,. \tag{4.123}$$
- Upon a Weyl transformation therefore, ANY 2d surface has a flat metric.
- In 2d Einstein tensor vanishes and hence 2d Einstein gravity is ill-defined.
- **Orientibility:** Any 2d surface can be embedded in 3d flat space and using the embedding one can easily define an orientation on the 2d surface, especially if the surface is periodic in one or two directions.
- **Topology:** Given the metric, the manifold may admit various global structures and topologies: the manifold can be compact, have a finite volume and no boundary; or non-compact has boundary and/or infinite volume.
- The topological information of 2d manifolds are defined by their *genus* and number of boundaries.
- The Euler character,
$$\chi \equiv \frac{1}{2\pi}\int d^2z\sqrt{g}\mathcal{R} \,,$$
classifies the geometries. Genus $g = 1-(\chi+b)/2$, where b is number of boundaries, defines number of handles.
▸▸ Exercise 4.69: *Show that χ does not change under Weyl scaling.*
- For a sphere χ is 2, for a torus it is zero. Higher genus surfaces should hence come from (uniformly) negative curved surfaces upon orbifolding, explicitly, a torus is $R^2/\mathbb{Z}^2$ and higher g are H^2/Γ.

• If it were not for global issues, it would have been enough to study 2d CFTs on R^2 or on a part of R^2 (if we have boundary).

▸▸ Exercise 4.70: ***Linear dilaton theory:*** *Consider an N scalar model with the action*
$$S = \frac{1}{4\pi}\int_\Sigma d^2z\ (\partial\phi_a\bar{\partial}\phi_a + V^a\phi_a\ \mathcal{R}) \,, \tag{4.124}$$
where V^a is ia constant given N-vector and $\mathcal{R}$ is the curvature of the 2d surface Σ.
I. *Quantize the theory and compute its spectrum.*
II. *Compute energy momentum T of this theory.*
III. *Compute the central charge of the theory and show that it is $c = N + 6V_aV^a$.*
Hint: *The last term may be simplified recalling* (4.123).

4.10.2 Different 2d tori

- A 2d Torus can be obtained from an R^2 via identification of the two independent directions.
- The directions over which we make the identification can be orthogonal or not. Without loss of generality, one can choose one of the identification directions as x^1 axis (real axis) with radius R_1, and then other one will be specified by a complex number τR_1, where τ is a complex number, such that $R_2 = |\tau R_1|$ and its angle with x^1 axis is α. The volume (area) of the torus is hence $4\pi^2 R_1 R_2 \sin\alpha = (2\pi R_1)^2 Im\tau$.
- Any torus of given R_1 or area is then specified by a complex number τ and its metric is given as

$$ds^2 = R_1^2 |dx + \tau dy|^2\,, \qquad x \equiv x + 2\pi\,,\ y \equiv y + 2\pi\,, \quad \tau = \frac{R_2}{R_1} e^{i\alpha}\,. \tag{4.125}$$

- τ is usually called **modular** parameter, or **shape** parameter or **complex structure** parameter of the torus.
- Due to topological obstruction resulting from periodicity of the axes not all tori with any given τ can be related to each other by a rotation in y axis. The question is which tori are *inequivalent*.
- The above coordinate frame, specifies a lattice where a generic point on it is given by $m + n\tau,\ m, n \in \mathbb{Z}$. A torus of the same **shape** and same area is formed if instead of $(0, 1)$ and $(1, 0)$ points, the identification is done along $(m+n\tau)$ and $(p+q\tau)$. In order this torus to have the same area as before we need to require $mq - np = 1$. The new torus obtained in this way will hence have

$$\tau' = \frac{m + n\tau}{p + q\tau}\,, \qquad mq - np = 1\,, m, n, p, q \in \mathbb{Z}\,. \tag{4.126}$$

- The tori whose modular parameters are related by an element of the *modular transformation group* $PSL(2, \mathbb{Z}) = SL(2, \mathbb{Z})/\mathbb{Z}_2$ (the extra $\mathbb{Z}_2$ changes sign of all integers m, n, p, q) are hence equivalent.
- The space of all possible complex numbers τ which upon an $SL(2, \mathbb{Z})$ transformation covers the whole 2d plane is called **fundamental domain**.

  ▶▶ **Exercise 4.71:** *Draw the fundamental domain.*
- Each point in the fundamental domain will hence specify an inequivalent torus.

4.10.3 Partition function of 2d CFT on torus

NOTE: *The arguments in section are taken from the paper F. Loran, **M.M.Sh-J**, M. Vincon,, **JHEP 1101 (2011) 110**, `arXiv:1010.3561`.*

- Consider a generic 2d CFT with central charge $\mathbf{c}$ and the Virasoro algebra

$$[L_n, L_m] = (n-m)L_{m+n} + \frac{\mathbf{c}}{12}n^3\delta_{m+n} \qquad [\bar{L}_n, \bar{L}_m] = (n-m)\bar{L}_{m+n} + \frac{\bar{\mathbf{c}}}{12}n^3\delta_{m+n}$$

  NOTE: *The above algebra is written for conformal generators on the cylinder (and not the plane). That is, we are working with*

$$L_n = L_n^{cyl} = L_n^{plane} - \frac{\mathbf{c}}{24}. \tag{4.127}$$

- A few words on **non-zero temperature QFT at equilibrium.**
  - There are different ways of describing a non-zero temperature QFT at equilibrium, a standard way is to consider the Euclidean version of the QFT and compactify the Euclidean time direction on a circle of radius β.
  - Partition function of the theory may be computed using path integral in Hamiltonian formulation:

$$\begin{aligned} Z_{QFT}(\beta) &= \text{Tr}\ \left(e^{i\int Hdt}\right) \\ &= \text{Tr}\ \left(e^{-2\pi\beta H_{\text{Eucl.}}}\right)\ , \end{aligned} \tag{4.128}$$

    where the Tr is over all physical states in the Hilbert space. In writing the second line, we have assumed that 1) the theory is in equilibrium and 2) the partition functions of the Minkowski and Euclidean theories are the same.
  - Had we other conserved charges in the system like Q_i one can compute partition function of the theory over the subset of the Hilbert space with states of a given Q_i. As is standard practice in stat.mech. systems, one may then compute the partition function in this sector by adding Q_i to the Hamiltonian with a chemical potential μ_i while still taking the trace over all states in physical Hilbert space:

$$Z_{QFT}(\beta;\mu_i) = \text{Tr}\ \left(e^{-2\pi\beta H_{\text{Eucl.}}+2\pi\sum_i \mu_i Q_i}\right)\ , \tag{4.129}$$

    the factor of 2π is just by convention.
- To heat up a given 2d CFT on cylinder, we should hence compactify the time direction, which is along the axes of the cylinder on a circle. This of course puts the CFT on a torus. That is, a 2d CFT on the torus is nothing but a 2d CFT at non-zero temperature.
- We compute the partition function at a given temperature and a given chemical potential for the angular momentum, conjugate to Hamiltonian E and angular momentum J. Instead, one may consider $L_0,\ \bar{L}_0$ as sum and difference of E and J.
- The partition function will hence depend on temperature inverse β and the angular momentum chemical potential, say ω. These are basically the radii of the two cycles of the torus.

- At a more technical level this partition function, being the partition function of a CFT, should only depend on the **shape** parameters and not **size** parameters of the torus. That is, partition function should be a function of the modular parameter τ.

- Therefore, partition function of the above theory on the a torus with modular parameter τ is

$$Z(\tau,\bar{\tau}) = \text{Tr}\ \left(e^{2\pi i\tau L_0}\ e^{-2\pi i\bar{\tau}\bar{L}_0}\right) \tag{4.130}$$

  where the Tr is over all states in the Fock space of the theory, it includes summation over the primaries as well as the states in their Verma modules, and $\bar{\tau} = \tau^*$.

- Recalling that L_0 and $\bar{L}_0$ are essentially Hamiltonians of the left (holomorphic) and right (anti-holomorphic) sectors of 2d CFT on the torus, the form of partition function is that of a non-zero temperature QFT, with **left and right temperatures,** respectively T_L and T_R, where

$$\beta_L \equiv 1/T_L = -2\pi i\tau\,, \qquad \beta_R \equiv 1/T_R = +2\pi i\bar{\tau}\,, \tag{4.131}$$

  NOTE: *The above temperatures are not real-valued. However, there is a "physical" real temperature associated with total radial Hamiltonian $L_0+\bar{L}_0$. This physical temperature T is hence*

$$T = \frac{1}{\beta}\,, \qquad \beta = \beta_L + \beta_R\,, \tag{4.132}$$

  ►► **Exercise 4.72:** *Investigate and work out the above.*

- For a unitary theory, which we assume here, the trace is over all the states and since all states have a positive norm their contribution add up. (Formally, for a non-unitary theory the positive norm states contribution is subtracted from the negative norm state contributions.)

- Partition function $Z(\tau,\bar{\tau})$ is expected to be invariant under $PSL(2,\mathbb{Z})$ modular transformations. This is due to the fact that a torus is defined up to modular transformations, *i.e.*

$$Z(\tau',\bar{\tau}') = Z(\tau,\bar{\tau})\,, \qquad \tau' = \frac{a\tau+b}{c\tau+d}\,, \quad \bar{\tau}' = \tau'^*\,, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{Z})\,. \tag{4.133}$$

- Let's denote the eigenvalues of L_0 and $\bar{L}_0$ (Hamiltonians on the torus) by $\Delta,\ \bar{\Delta}$. We argued in the previous section that unitarity requires eigenvalues of L_0^{plane}, denoted by conformal weight h, to be positive. Therefore,

$$\Delta \equiv h - \frac{\mathbf{c}}{24} \geq -\frac{\mathbf{c}}{24}\,, \qquad \bar{\Delta} \equiv \bar{h} - \frac{\bar{\mathbf{c}}}{24} \geq -\frac{\bar{\mathbf{c}}}{24}\,. \tag{4.134}$$

  NOTE: *Δ is eigenvalue of L_0 on cylinder, while h is eigenvalue of L_0 on the plane.*

- Noting that all states in the Fock space of a unitary CFT have definite $h, \bar{h}$, the Tr is replaced with a sum over $h, \bar{h}$:

$$Z(\tau,\bar{\tau}) \equiv \sum_{h,\bar{h}} \rho(h,\bar{h})\; e^{2\pi i\tau(h-\mathbf{c}/24)}\; e^{-2\pi i\bar{\tau}(\bar{h}-\bar{\mathbf{c}}/24)} \tag{4.135}$$

where $\rho(h,\bar{h})$ is density of states with a given $h, \bar{h}$. Note that this includes both primaries and their descendents.

NOTE: *To write the above,* ***Unitarity*** *has been assumed, assuming that ρ is positive definite. Explicitly, if we dealt with a non-unitary theory we would not have been able to extract density of all states, but just the difference between the positive norm and negative norma densities.*

NOTE: *In* (4.135) *we have also assumed that $L_0, \bar{L}_0$ have a* ***discrete spectrum****.*

- Our aim is to use modular invariance (4.133) and compute the *density of states* $\rho(h,\bar{h})$.

- To compute ρ we introduce

$$q = e^{2\pi i\tau}\,, \qquad \bar{q} = e^{-2\pi i\bar{\tau}}\,, \tag{4.136}$$

and treat τ and $\bar{\tau}$ and, hence q and $\bar{q}$, as independent complex variables.

- Next, use the contour integrals over $q,\ \bar{q}$ to compute ρ:

$$\begin{aligned}\rho(h,\bar{h}) &= \frac{1}{(2\pi i)^2}\int \frac{dq}{q^{h+1}}\frac{d\bar{q}}{\bar{q}^{\bar{h}+1}}\; Z(q,\bar{q})q^{\mathbf{c}/24}\bar{q}^{\bar{\mathbf{c}}/24}\\ &= \frac{1}{(2\pi i)^2}\int d\tau d\bar{\tau}\; Z(\tau,\bar{\tau})e^{-2\pi i\tau(h-\mathbf{c}/24)}\; e^{2\pi i\bar{\tau}(\bar{h}-\bar{\mathbf{c}}/24)}\,.\end{aligned} \tag{4.137}$$

NOTE: *To compute density of states $\rho(h,\bar{h})$ using the contour integrals we have assumed that partition function is analytic function of a power of $q, \bar{q}$. This latter in fact follows from the discreteness of the spectrum condition we assumed.*

- We now use **modular invariance** (4.133) to constrain and compute ρ. Let us consider the S-transformation

$$\tau \;\to\; -\frac{1}{\tau} \qquad q \;\to \hat{q} = e^{-2\pi i/\tau}\,.$$

We then learn that

$$\sum_{h,\bar{h}} \rho(h,\bar{h})q^{h-\mathbf{c}/24}\bar{q}^{\bar{h}-\bar{\mathbf{c}}/24} = \sum_{h',\bar{h}'} \rho(h',\bar{h}')\hat{q}^{h'-\mathbf{c}/24}\hat{\bar{q}}^{\bar{h}'-\bar{\mathbf{c}}/24} \tag{4.138}$$

- One can then use (4.138) to obtain a recursion relation for ρ

$$\rho(h,\bar{h}) = \sum_{h',\bar{h}'}\rho(h',\bar{h}')\; I(\Delta,\Delta')I(\bar{\Delta},\bar{\Delta}')\,, \tag{4.139}$$

where

$$I(\Delta,\Delta') = -\int_0^{i\infty(+)} d\tau\ e^{-2\pi i(\Delta\tau - \frac{\Delta'}{\tau})} = (-i)\int_{-\infty}^{0(+)} \frac{d\tau}{\tau^2}\ e^{-2\pi(\Delta\tau + \frac{\Delta'}{\tau})} \tag{4.140}$$

where Δ and h are related as in (4.134).

- The above integral is a standard integral representation of Bessel function. Assuming $\Delta \geq 0$, we have:

$$I(\Delta,\Delta') = \begin{cases} -2\pi\sqrt{\frac{\Delta'}{\Delta}}\ I_1(4\pi\sqrt{\Delta\Delta'})\,, & \Delta' > 0 \\ \\ 2\pi\sqrt{\frac{|\Delta'|}{\Delta}}\ J_1(4\pi\sqrt{\Delta|\Delta'|})\,, & \Delta' < 0 \end{cases} \tag{4.141}$$

NOTE: *Bessel functions of I and J type are related to each other, roughly as $I(iz) \sim J(z)$.*

- Let us assume that $\Delta, \bar{\Delta}$ are positive, while $\Delta', \bar{\Delta}'$ over which we have sums can be positive and negative. Therefore, in the RHS of (4.139) we will have four terms:

$$\begin{aligned} \rho(h,\bar{h}) = (2\pi)^2 &\sum_{\Delta'\leq 0,\bar{\Delta}'\leq 0} \rho(\Delta',\bar{\Delta}')\ \frac{|\Delta'|}{U} I_1(4\pi U)\frac{|\bar{\Delta}'|}{V} I_1(4\pi V) \\ - (2\pi)^2 &\sum_{\Delta'\leq 0,\bar{\Delta}'>0} \rho(\Delta',\bar{\Delta}')\ \frac{|\Delta'|}{U} I_1(4\pi U)\frac{|\bar{\Delta}'|}{V} J_1(4\pi V) \\ - (2\pi)^2 &\sum_{\Delta'>0,\bar{\Delta}'\leq 0} \rho(\Delta',\bar{\Delta}')\ \frac{|\Delta'|}{U} J_1(4\pi U)\frac{|\bar{\Delta}'|}{V} I_1(4\pi V) \\ + (2\pi)^2 &\sum_{\Delta'>0,\bar{\Delta}'>0} \rho(\Delta',\bar{\Delta}')\ \frac{|\Delta'|}{U} J_1(4\pi U)\frac{|\bar{\Delta}'|}{V} J_1(4\pi V) \end{aligned} \tag{4.142}$$

where

$$\begin{aligned} U &\equiv \sqrt{|\Delta'|\Delta} = \sqrt{|h' - \mathbf{c}/24|(\bar{h} - \mathbf{c}/24)}\,, \\ V &\equiv \sqrt{|\bar{\Delta}'|\bar{\Delta}} = \sqrt{|\bar{h}' - \bar{\mathbf{c}}/24|(\bar{h} - \bar{\mathbf{c}}/24)}\,. \end{aligned} \tag{4.143}$$

NOTE: *We are already assuming that $\Delta = h - \mathbf{c}/24 \geq 0$, $\bar{\Delta} = \bar{h} - \bar{\mathbf{c}}/24 \geq 0$.*

- Recalling that for large $z \gg 1$

$$J_1(z) \sim \sqrt{\frac{2}{\pi z}}\sin(\frac{\pi}{4} - z)\,, \qquad I_1(z) \sim \sqrt{\frac{1}{2\pi z}} e^z\,, \tag{4.144}$$

the last three lines in (4.142)is much smaller than the first line which involves Bessel-I function.

NOTE: *The sums in last three lines involve infinite number of terms (e.g. recall the states in the Verma module. Nonetheless, all these terms come with the "fast" oscillatory term of Bessel-J and hence cancel off each other. Although we cannot show that they are equal to zero for any given 2d CFT with an arbitrary spectrum, one can safely ignore them up to* ***exponentially suppressed terms*** *compared to the first line.*

- Therefore, up to exponentially suppressed corrections,

$$\rho(\Delta, \bar{\Delta}) = (2\pi)^2 \sum_{\Delta' \le 0, \bar{\Delta}' \le 0} \rho(\Delta', \bar{\Delta}') \, \frac{|\Delta'|}{U} I_1(4\pi U) \frac{|\bar{\Delta}'|}{V} I_1(4\pi V) \tag{4.145}$$

- Finally, let us recall the exponential behavior of Bessel-I *cf.* (4.144) and the assumption that our theory has a discrete spectrum. Therefore, the term with lowest $\Delta', \bar{\Delta}'$, which has largest U, V values for given $\Delta, \bar{\Delta}$, will have the largest contribution to the above sum; the other terms will be exponentially suppressed compared to this term. That is, up to exponentially suppressed terms:

$$\rho(\Delta, \bar{\Delta}) = \left(\frac{\pi^2}{3}\right)^2 \rho_0 \, \hat{\mathbf{c}} \, \frac{I_1(S_0)}{S_0} \cdot \hat{\bar{\mathbf{c}}} \, \frac{I_1(\bar{S}_0)}{\bar{S}_0} \, . \tag{4.146}$$

In the above

  - $\rho_0 = \rho(\Delta_0, \bar{\Delta}_0)$ is the density of states in "ground state", the vacuum state, which comes with conformal weights $h_0 = \Delta_0 + \frac{\mathbf{c}}{24}, \bar{h}_0 = \bar{\Delta} + \frac{\bar{\mathbf{c}}}{24}$. ρ_0 may be taken to be one.
  - $\hat{\mathbf{c}}$, $\hat{\bar{\mathbf{c}}}$ are **effective central charges** are equal to

$$\hat{\mathbf{c}} = \mathbf{c} - 24\Delta_0 \, , \qquad \hat{\bar{\mathbf{c}}} = \bar{\mathbf{c}} - 24\bar{\Delta}_0 \, . \tag{4.147}$$

  - expression S_0 and $\bar{S}_0$ are

$$S_0 = 2\pi\sqrt{\frac{\hat{\mathbf{c}}}{6}(\Delta - \frac{\mathbf{c}}{24})} \, , \qquad \bar{S}_0 = 2\pi\sqrt{\frac{\hat{\bar{\mathbf{c}}}}{6}(\bar{\Delta} - \frac{\bar{\mathbf{c}}}{24})} \tag{4.148}$$

NOTE: *Equation* (4.146) *provides density of states for modes with positive* $\Delta, \bar{\Delta}$.

- As we see up to exponentially suppressed terms the density of states is product of the contribution from left movers and that of right movers. Note that, this is not the case in general, before ignoring the exponentially suppressed terms in (4.142) and (4.145).

- One may insert the above density of states in (4.135) and compute the partition function Z.

▶▶ **Exercise 4.73:** *Show that the partition function will be "holomorphically factorized",* i.e. *it takes the form of*

$$Z(\tau, \bar{\tau}) = Z(\tau) \cdot \bar{Z}(\bar{\tau}) \, , \tag{4.149}$$

where

$$Z(\tau) = \frac{1}{12}\left(e^{\frac{\pi i \hat{\mathbf{c}}}{12\tau}} - 1\right), \qquad \bar{Z}(\bar{\tau}) = \frac{1}{12}\left(e^{-\frac{\pi i \hat{\bar{\mathbf{c}}}}{12\bar{\tau}}} - 1\right). \tag{4.150}$$

NOTE: *The above expressions hence give the partition function of the theory up to exponentially suppressed corrections. Note that in computing this partition function we only used* ***unitarity, discreteness of spectrum and modular invariance****. This is the miracle of conformal symmetry, no details of the theory and its interactions has been used.*

- One may take logarithm of the density of states to obtain the **entropy** associated with the 2d CFT on the torus. Assuming that $S_0 \gg 1,\ \bar{S}_0 \gg 1$, in the leading order we obtain

$$S_{Cardy} = S_0 + \bar{S}_0 = S_0 = 2\pi\left(\sqrt{\frac{\hat{\mathbf{c}}}{6}(h - \frac{\mathbf{c}}{24})} + \sqrt{\frac{\hat{\bar{\mathbf{c}}}}{6}(\bar{h} - \frac{\bar{\mathbf{c}}}{24})}\right) \tag{4.151}$$

where $\hat{\mathbf{c}},\ \hat{\bar{\mathbf{c}}}$ are effective central charges defined in (4.147).

NOTE: *Eq.*(4.151) *is the celebrated* ***Cardy formula*** *for the entropy of any 2d CFT. This is very remarkable, as it only depends on the effective central charge and the energy level the system is excited to, and not on the details of the theory.*

▶▶ **Exercise 4.74:** *Using* (4.146) *and the explicit form of the Bessel-I function, compute subdominant contribution to the entropy.*

▶▶ **Exercise 4.75:** *Using the expression for the partition function* (4.149) *compute free energy and the entropy of the 2d CFT system as a function of temperatures* $T_L,\ T_R$ cf. (4.131), *using standard thermodynamical equations and show that*

$$S = \frac{\pi^2}{3}(\hat{\mathbf{c}} T_L + \hat{\bar{\mathbf{c}}} T_R)\,, \tag{4.152}$$

where T_L, T_R *are defined in* (4.131).
The entropy given in (4.152) *is hence the entropy computed in* ***canonical*** *formulation while the one computed in previous exercise is the entropy in the* ***micro-canonical*** *formulation. Compare the two.*

4.11 2d CFT on a generic 2d surface and trace anomaly

- As argued any 2d surface, up to possibly topological obstructions, can be brought to 2d flat space. As such, we mainly focused on the 2d CFT's on (locally) flat 2d spaces. The question we pose in this section is whether the conformal map (i.e. Weyl scaling) which brings a generic non-flat surface to a flat one is anomalous or not.

- To address this question we recall (*cf.* discussions of section 4.3) that invariance under *rigid scaling* leads to tracelessness of the energy-momentum tensor, i.e. $T_{\bar{z}z} = T_{z\bar{z}} = 0$. Then the question is

$$\delta\langle T_{\bar{z}z}\rangle \overset{?}{=} 0\,. \tag{4.153}$$

With the above of course we mean $\delta\langle T_{\bar{z}z}\mathbf{X}\rangle \overset{?}{=} 0$ for any arbitrary insertion $\mathbf{X}$.

- To answer this question we take two steps:
   i. Consider a map from flat space to an arbitrary curved space:

$$w \to z = z(w) \quad \text{where} \quad dwd\bar{w} \to |\frac{dz}{dw}|^2 dwd\bar{w}\,. \tag{4.154}$$

   and compute $\delta_w\langle T_{\bar{z}z}\rangle$.

   ii. Assume translation invariance to be *non-anomalous*, use (4.15) and (4.17) to relate $T_{\bar{z}z}$ to $T(z)$.

- To implement the first step we note that $T_{\bar{z}z} = \frac{1}{4\pi}\frac{\delta S}{\delta g^a_a}$ and that

$$\begin{aligned}
\delta_w\langle T_{\bar{z}z}(z)\rangle &= \delta_w\left(\int D\Phi e^{-S[\Phi]}\ T_{\bar{z}z}\right) \\
&= \int D\Phi e^{-S[\Phi]}\ \left(\int d^2w\sqrt{g}\ T_{\bar{z}z}\delta_w g^{ab}T_{ab}\right) \\
&= \frac{1}{2\pi}\int d^2w\langle T_{\bar{z}z}T_{\bar{w}w}\rangle\Omega(w,\bar{w})\,.
\end{aligned} \tag{4.155}$$

   where in the last line we focused on the maps (4.154) for which $\delta_w g^{ab} = \Omega(w,\bar{w})\delta^{ab}$.

- We then use translation invariance (the assumption in step ii. above)

$$\partial T_{\bar{z}z} = \bar{\partial}T(z)\,,$$

   the standard $T(z)T(w)$ OPE (4.42), that w-plane is flat space for which $\langle T(w)\rangle = 0$, and (4.123) to arrive at

$$\langle T_{\bar{z}z}\rangle = -\frac{\mathbf{c}}{12}\mathcal{R}\,, \tag{4.156}$$

   where $\mathcal{R}$ is the curvature of the 2d surface spanned by z-coordinates.

   ►► **Exercise 4.76:** *Fill out possible computational gaps in the above.*

- One could have repeated the same analysis for $T_{z\bar{z}}$ to find

$$\langle T_{z\bar{z}}\rangle = -\frac{\bar{\mathbf{c}}}{12}\mathcal{R}\,. \tag{4.157}$$

- As we see, IF $\mathbf{c} = \bar{\mathbf{c}}$ then the above two relations are consistent. This means that the assumption of translation symmetry being non-anomalous is consistent. However, if $\bar{\mathbf{c}} \neq \mathbf{c}$ then the translation symmetry is also anomalous.

- Moreover, the above shows that 2d CFT's on flat space the trace anomaly vanishes.

- Eqs.(4.156) and (4.157) imply that 2d conformal symmetry can be **gauged** iff $\mathbf{c} = \bar{\mathbf{c}} = 0$, and that, 2d gravity is well-defined only if $\mathbf{c} = \bar{\mathbf{c}}$; if $\bar{\mathbf{c}} \neq \mathbf{c}$ we have 2d gravitational anomaly.

- The above gives another interpretation for the central charge $\mathbf{c}$: It appears in the trace anomaly (Weyl scaling anomaly) relation.

- Noting that in 2d the only diff. invariant object of dimension of two is the Ricci scalar, and that the anomaly relation should be generally invariant one could have guessed that $\langle T_{z\bar{z}}\rangle \propto \mathcal{R}$. The coefficient, which is a property of the theory in question, is $-\mathbf{c}/12$. We will return to this point later on when we discuss 4d CFT's.

►► **Exercise 4.77:** *Compute $\langle T_{\bar{z}z}(z,\bar{z})T(w)\rangle$, $\langle T_{\bar{z}z}(z,\bar{z})T(w)T(u)\rangle$ and $\langle T_{z\bar{z}}(z,\bar{z})\bar{T}(\bar{w})\bar{T}(\bar{u})\rangle$. The anomaly expression may also be recast in terms of the last two.*

4.12 The c-theorem

- As discussed in the case of free multi boson (or multi fermion) theory, and also as is apparent from the Cardy formula (4.152) the central charge is a measure of number of degrees of freedom of a 2d CFT.

- We know that CFT's may be viewed as usual QFT's at their RG fixed point. This is more robust for the 2d case where it is proven that (*cf.* discussions of section 3.4.1) upon some reasonable conditions scale invariance (which is recovered at the RG fixed point) implies conformal invariance.

- Conversely, one may view a generic 2d QFT as a deformation around a 2d CFT. This deformation can be by relevant, marginal and irrelevant local operators.

- **Zamolodchikov's c-theorem** in 2d QFT's states that

  On the space of QFTs, there exists a functional $c(\mu;\lambda_i)$, μ being the RG scale and λ_i are couplings. Under RG flow to IR the **c***-function decreases monotonically and at RG fixed points, where the QFT reduces to a CFT,* **c***-function evaluated at this fixed point becomes the central charge of the corresponding CFT.*

- The 2d $\mathbf{c}$-theorem was proved in
  `A. B. Zamolodchikov, ''Irreversibility of the Flux of the Renormalization Group in a 2D Field Theory,'' JETP Lett.` **43** `(1986) 730 [Pisma Zh. Eksp. Teor. Fiz.` **43** `(1986) 565].`

  NOTE: *Once at a fixed point in the parameter space specified by couplings λ_i^*, the value of* **c***-function does not depend on the energy scale μ, is only a function of the values of the couplings at the fixed point.*

- The $\mathbf{c}$-function represents the number of degrees of freedom in the QFT, or the CFT arising in the fixed points.

- We may start from a CFT at UV, deform it by some relevant or marginal operator to arrive at a different CFT, with a different central charge in IR. Then, the $\mathbf{c}$-function interpolates between the two CFT's. In this case $\mathbf{c}$-theorem states that $\mathbf{c}_{IR} < \mathbf{c}_{UV}$.

5 4d CFT's

In sections 2 and 3 we discussed conformal group in general dimensions, the conditions coming from unitarity of representations, general structure of two, three, four and higher point functions, and the Ward identities associated with conformal symmetry. In this section we discuss some other aspects of 4d CFT's. However, let us first give a brief review of our earlier discussions. Then, give a motivation for 4d CFT's which are generically of the form of Yang-Mills gauge theories, and are also usually supersymmetric theories.

5.1 Review of our earlier results

- Conformal algebra in 4d is $SO(4,2)$ for Lorentzian signature and $SO(5,1)$ for Euclidean.
- Its unitary reps are labelled by the scaling dimension Δ and their spin.
- The Lagrangian of scale invariant QFT's on $\mathbb{R}^4$ cannot involve any dimensionful coupling/parameter. In particular, it should only involve massless states.
- In the Euclidean signature, CFT's on $\mathbb{R}^4$ in the radial quantization are related to the same theory on $R_\tau \times S^3$ with translation along τ direction related to radial Hamiltonian.

  NOTE: *Recall that on $R\times S^3$ a CFT theory can involve a mass parameter, a conformal mass term,* cf. ***Exercise 3.3***.
- The conformal representations are specified by (quasi)primary states which transform as (2.18), if we use operator language. Alternatively, one may use operator-state correspondence and discuss about primary states. Primary states are states with given spin and scaling dimension Δ which are killed by the action of special conformal transformations generator K_μ. Explicitly,

$$K_\mu|Primary\rangle = 0\,.$$

- The other states in the same conformal multiplet are then constructed by the action of P_μ and other $SO(4,2)$ generators on the primary states. That is, we know the full spectrum of a CFT if we know the spectrum of its primaries. These states which again have definite scaling dimension and spin are called descendent of the primary. If a descendent is constructed by ℓ number of P_μ's acting upon a primary, we have a level ℓ descendent.
- **Unitarity bounds.** Unitarity of representation imposes bounds on the *scaling dimension of the primaries.* For example, demanding all primaries to have positive-definite norm leads to $\Delta \geq 0$.
- Unitarity, however, also demands all level ℓ descendents should have positive norm. In **four dimensions** this leads to the following spin dependent unitarity bounds and

conditions:

$$\begin{aligned}
\Delta \geq 1 &\qquad \text{for primary scalar fields,} \\
\Delta \geq 3/2 &\qquad \text{for primary spin 1/2 fields,} \\
\Delta \geq 3 &\qquad \text{for primary gauge field (currents),} \\
\Delta \geq 4 &\qquad \text{for symmetric traceless tensor (like the energy momentum tensor),} \\
\Delta \geq l+2 &\qquad \text{for primary spin} l \text{fields.}
\end{aligned} \tag{5.1}$$

NOTE: *Free massless field theories which are of course CFT's, saturate these bounds.*

NOTE: *For vector case, the primary rep. must also be divergence free. This implies that spin one primary states can be viewed as currents coupled to gauge fields.*

- The spacetime dependence of two and three point functions are completely fixed. For scalar (spin zero) operators, that is as given in (3.49) and (3.50).

  ▶▶ **Exercise 5.1:** *As in the 2d case,* cf. *discussions of section 4.9.1, the three point function of any three operators, primary or descendent, can be recast in terms of three point function of primaries. If the "vertex" function of three primary operators* $\mathcal{O}_i$, $\mathcal{O}_j$, $\mathcal{O}_k$ *is denoted by* C^p_{ijk}. *Compute the vertex function of three generic operators. Note that 4d conformal multiplets has a much simpler structure than the 2d Verma modules.*

- The spacetime dependence of four and higher point functions is specified up to dependence on conformal cross-ratios, e.g. as in (3.51) for scalar spin-zero operators.

  ▶▶ **Exercise 5.2:** *As in the 2d case,* cf. *discussions of section 4.9.2, show that one may use OPE of primary operators to reduce the four point function of* primary operators *to a sum over products of two* $C_{12p}C_{34p}$ *for primaries, plus a kinematical information encoded in* conformal blocks*, as in* (4.119) *and* (4.120).

5.2 Examples of 4d CFTs

- In the previous subsection we reviewed basic properties of 4d CFT's mainly on $\mathbb{R}^4$. However, one would like to know if there are non-trivial (interacting) 4d QFT's which exhibit conformal invariance, both at classical level and quantum mechanically.

- Scale invariance for a perturbative 4d QFT at classical level implies that we should only consider massless fields with deformations by marginal local Lorentz invariant operators. These will come with dimensionless couplings λ_i^{marg}.

- In general the β-function of these couplings is non-zero and hence scale invariance will be lost at quantum level; CFT's can only appear in RG fixed points.

- At these fixed points, however, the theory may not be/remain perturbative, or may not necessarily admit a simple Lagrangian description.

  NOTE: *At fixed points the theory is scale invariant and not necessarily conformal invariant. Recall discussions of section 3.4.1.*

- A candidate 4d CFT which admits Lagrangian description may be sought for in the family of Yang-Mills gauge theories at their fixed points. In fact, all known examples of (perturbatively accessible) 4d CFTs are in the family of gauge theories.

- Let us start with 4d $SU(N_c)$ Yang-Mills theory plus N_f *massless Dirac* fermion flavors in fundamental rep. of $SU(N_c)$. This is a QCD-like theory.

- In the UV the theory can be Asymptotic Free (AF) if the β-function of the gauge coupling is negative. If so, the theory has a perturbatively accessible UV-fixed point. At this fixed point, while a CFT, the theory is free and hence not so interesting.

- The question is then, if this theory has also non-trivial IR fixed points. To address this question we should study zeros of the β-function which is

$$\beta(g) = \sum_{\ell=1} \beta_\ell g^{2\ell+1}\,, \qquad \beta_\ell = \sum_{m=0}^{\ell} A_{\ell m} N_c^m N_f^{\ell-m}\,, \tag{5.2}$$

where β_ℓ is the ℓ-loop β-function and $A_{\ell m}$ are some numeric coefficients which may have dependence on N_c with negative powers of N_c and are computable by explicit loop calculations. The one and two loop results are

$$A_{11} = -\frac{1}{16\pi^2}\frac{11}{3}\,, \qquad A_{10} = \frac{1}{16\pi^2}\frac{2}{3}\,,$$
$$A_{22} = -\frac{1}{16\pi^2}\frac{34}{3}\,, \qquad A_{21} = +\frac{1}{16\pi^2}(\frac{10}{3} + \frac{N_c^2-1}{N_c^2})\,, \qquad A_{20} = 0\,. \tag{5.3}$$

▶▶ **Exercise 5.3:** *Show that β-function can be written as*

$$\beta(\lambda) = 2\sum_{\ell=1}\sum_{m=0}^{\ell} A_{\ell m}\left(\frac{N_f}{N_c}\right)^{\ell-m} \lambda^{\ell+1}\,, \tag{5.4}$$

where $\lambda = g^2 N_c$ is the 't Hooft coupling.
NOTE: *'t Hooft coupling provides a particularly nice expansion for large N_c, or when $N_f/N_c \ll 1$. Moreover, in this limit $A_{\ell m}$ coefficients become only some numbers and their N_c dependence drops out.*

- Therefore, if

$$\beta_1 \leq 0 \quad \Rightarrow \quad N_c \geq 2N_f/11 \tag{5.5}$$

and hence the theory is AF.

- For higher loops we know that

$$A_{\ell\ell} < 0\,, \qquad A_{\ell 0} \geq 0. \tag{5.6}$$

Therefore, for large N_c with finite N_f, β_ℓ will be dominated by the N_c^ℓ term and always remains negative, with no zeros, while for large N_f with finite N_c, β-function is dominated by the flavor terms and is hence always positive.

- Depending on the values of N_f, N_c, higher loop contributions to β-function may become positive and can in principle dominate over the one-loop result. This brings up the possibility of β-function having zeros.
- Let us e.g. focus up to two loop results.
  - To have asymptotic freedom we need $\beta_1 \leq 0$,
  - to have (IR) fixed points we need $\beta_2 \geq 0$, moreover to remain perturbative we need $|\beta_1/\beta_2| \lesssim 1$.
- It is indeed possible to find N_f, N_c which satisfy the above conditions and hence we have a perturbative IR fixed point. These values of N_f and N_c provide a *conformal window*. This fixed point is usually called **Banks-Zaks fixed point**.

  ▶▶ **Exercise 5.4:** *Do we have an IR fixed point for any given value of N_c? If yes, assuming we can trust the perturbative β-function result* (5.2)*, compute the range for N_f which specifies the conformal window in terms of $A_{\ell m}$ coefficients and the value of the coupling at the fixed point g_*.*

  ▶▶ **Exercise 5.5:** *Compute the value of the YM coupling g_* and the corresponding 't Hooft coupling λ_* at the IR fixed point.*

  ▶▶ **Exercise 5.6:** *Does having positive β_2 such that it can cancel off the one loop result destroy asymptotic freedom of the theory?!*
- Let us summarize the necessary conditions for existence and the possibility to arriving at Banks-Zaks fixed point:
  - *Negative β_1:* In order to define any QFT we need to start from a weakly coupled fixed point. Here we choose to start in the UV. As such, and to have a UV fixed point, we need to have an asymptotic free theory which is implied by $\beta_1 < 0$ condition.
  - *Positive β_2:* This is the necessary (but not sufficient) condition for having another (IR) fixed point.
  - $|\beta_2/\beta_1| \geq 1$*:* This condition is needed for two reasons: 1) To have a trustable perturbative loop expansion, upon which our analysis is based and, 2) To make sure that higher loop effects can be safely ignored. This latter of course needs further investigations.
  - *Ir fixed point should be attractor.* This latter is obviously needed if we want the RG flow to naturally land us on the IR fixed point, starting with relevant deformations in the UV. This condition implies that at the IR, Banks-Zaks, fixed point we should not have any **relevant** operator; deformations by relevant operators (by definition) takes us away from the fixed point and make the fixed point unstable.
- Some side remarks:
  - A confining theory just by definition cannot be conformal, because confinement means creation of a mass gap. Real QCD does not hence have a Banks-Zaks fixed point.

- For QCD-like theory with $N_c = 3$, there are computations (mainly lattice gauge theory results) indicating that the theory is confining with chiral symmetry breaking for $2 \leq N_f \leq 9$, for $10 \leq N_f \leq 12$ the theory is *perhaps* confining without chiral symmetry breaking, and for $13 \leq N_f \leq 16$ we have a conformal window. For $N_f > 16$ the theory is not asymptotic free.

  NOTE: *The above results are as of 2011 and to arrive at the above results one may not simply use QCD perturbation theory, as the coupling may be large.*
- For theories in the conformal window and at the IR fixed point global symmetry is the same as the UV theory, which is $U(N_f) \times U(N_f)$, each factor for one chirality of fermions.

  NOTE: *For more references on the above see papers by Slava Rychkov and references therein, may be found at his website:*
  `https://sites.google.com/site/slavarychkov/`.
- Note that in the conformal window and around the Banks-Zaks fixed point, we are not necessarily claiming that the perturbative degrees of freedom of theory are those of a gauge theory.

- One may ask if there are other 4d CFT's besides the YM+Dirac fund. fermion theories discussed above. The answer is of course positive. There are many more 4d CFT's which are mostly in the class of supersymmetric Yang-Mills theories. These are CFT's which have an action description.
- Superconformal gauge theories can come with various number of supersymmetries, $\mathcal{N} = 1,\ 2,\ 4$.
- In general, addition of SUSY imposes various cancelations in the loops and restricts the higher loop contributions to the *perturbative* β-function e.g. for $\mathcal{N} = 1$ β-function is two loop exact, for $\mathcal{N} = 2$ it is one-loop exact and for $\mathcal{N} = 4$ β-function is simply zero. This results are due to V. A. Novikov, M. A. Shifman, A. I. Vainshtein and V. I. Zakharov 1980's, and known as NSVZ β-function .
- The one loop perturbative β-function of any YM gauge theory with Weyl fermions in representation R_f of the gauge group and real scalars in representation R_s is:

$$\beta = -\frac{1}{16\pi^2} g^3 \left(\frac{11}{3} T(adj) - \frac{2}{3} \sum_f T(R_f) - \frac{1}{6} \sum_s T(R_s) \right) , \tag{5.7}$$

  where $T(rep)$ is the index for the corresponding representation. For $SU(N_c)$, $T(adj) = 2N_c$ and $T(fund) = 1$.
- $\mathcal{N} = 4$ **theory** is superconformal for any value of the gauge coupling, for any gauge group. That is, the theory has a conformal fixed line.
- Note that $\mathcal{N} = 4$ has 16 Poincaré supersymmetries, its smallest multiplet is the gauge multiplet which is the only multiplet $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) is comprised of. It consists of a gauge field, four chiral fermions and six real scalars,

all in the adjoint representation of the gauge group. (Note that there are eight real propagating boson and eight propagating fermion d.o.f in each multiplet.)

▶▶ **Exercise 5.7:** *Using the above information, that all the states in the same SUSY multiplet should be in the same gauge group reps, and* (5.7)*, show* β*-function of* $\mathcal{N} = 4$ *vanishes.*

- $\mathcal{N} = 2$ **theories** can be superconformal if the matter content is chosen such that the β-function vanishes. Since the β-function is one-loop exact, we again have a conformal fixed line, for any value of the coupling.

- Note that $\mathcal{N} = 2$ has 8 Poincaré supersymmetries. The $\mathcal{N} = 2$ SYM actions can consist of vector (gauge) multiplet and hypermultiplets. An $\mathcal{N} = 2$ vector multiplet consists of a gauge field, two Weyl fermions and a complex scalar (4 + 4 bosonic+fermionic propagating d.o.f) and a hypermultiplet contains two Weyl fermions and two complex scalars (again 4 + 4 bosons+fermions). Hypermultiplets can be in the fundamental, adjoint or other reps of the gauge group, while the fields in vector multiplet are all in the adjoint.

  ▶▶ **Exercise 5.8:** *Using* (5.7)*, show that for an* $SU(N_c)$ *theory with* N_h *hypermultiplets the perturbative* β*-function is*

$$\beta = -\frac{1}{4\pi^2} g^3 (N_c - X) , \qquad (5.8)$$

  where $X = N_c N_h$ *if the hyper is in the adjoint rep, and* $X = N_h/2$ *if the hyper is in fundamental rep.*

- Therefore, β-function vanishes if $N_h = 1$ with adjoint hyper and with $N_h = 2N_c$ for fundamental hyper. For these values we have a fixed line over which the theory is superconformal. With N_h larger than these values the theory is not asymptotic free.
  NOTE: $N_h = 1$ *with hyper in the adjoint is the same matter content of* $\mathcal{N} = 4$ *theory.*

- $\mathcal{N} = 1$ **theories** too, can be superconformal for appropriate matter content. Recalling that in $\mathcal{N} = 1$ we have vector multiplets containing a gauge field and a gaugino (which is a Weyl fermion) and chiral multiplets containing a chiral fermion and a complex scalar, one may use (5.7) to see for which matter content β-function vanishes. Whenever this happens we have a conformal fixed line.

  ▶▶ **Exercise 5.9:** *For a general* $\mathcal{N} = 1$ $SU(N_c)$ *theory with* N_{adj} *chiral multiplets in the adjoint and* N_{fund} *chiral fields in the fundamental, when does the* β*-function vanish?*

- Some side remarks,

  - $\mathcal{N} = 4$ SYM which is superconformal has Montonen-Olive S-duality symmetry. $\mathcal{N} = 2$ superconformal theories also enjoy S-duality.
  - Almost all of the known superconformal field theories (SCFT's) have a "gravity" dual, within AdS/CFT correspondence. That is, there is a supergravity theory

with appropriate number of supersymmetries with an AdS_5 vacuum solution. The details of the supergravity theory is in one-to-one correspondence with the matter content of the SCFT.

- $\mathcal{N} = 1, 2$ theories which are asymptotic free (negative β-function) can be confining and therefore, when flowing to IR other light degrees of freedom (meson-like) state can appear and hence change the behavior of the β-function (`Arkani-Hamed & Murayama '1997`). In particular, for $\mathcal{N} = 1$ SQCD, with $SU(N_c)$ gauge group with N_f fundamental matter (i.e. N_f chiral multiplets and N_f anti-chirals), when $3N_c/2 \leq N_f \leq 3N_c$ we have a conformal window and the theory flows to a strongly coupled CFT in the IR. (For $N_f > 3N_c$ the theory is IR free and the theory is not confining.)
- There are 4d SCFTs which are not YangMills theories, are always strongly coupled and presumably do not admit a Lagrangian description. For example, the $T_N,\ N = 3, 4, \cdots$ which are $\mathcal{N} = 2$ theories are in this class. Our handle on these theories is usually through AdS/CFT correspondence.

5.3 4d Superconformal algebras

- As discussed most of the better studied 4d CFT's (and also CFT's in general $d > 2$) are supersymmetric theories. SUSY brings protection of certain physical observables against quantum loop corrections, like mass of SUSY mesons, monopoles in confining theories and (perturbative) β-function .

- SUSY may be viewed in a quite algebraic view: Similarly to the conformal symmetry, super-Poincaré algebra is an extension of Poincaré symmetry in any dimension d. While conformal algebra is extension of Poincaré by dilation and special conformal transformation, super-Poincaré algebra is extension of Poincaré by addition of spinorial generators.

- There are many excellent review articles and books on super-Poincaré algebras, here we just review the basics and the interested reader may consult those references. As an example see, `P. West, arxiv:hep-th/9805055`.

5.3.1 Quick review of Super-Poincaré algebra

- Super-Poincaré algebra is constructed from Poincaré algebra upon addition of spinor generators, supercharges Q^I_a and $\bar{Q}_{I\dot{a}}$, where $I = 1, \cdots, \mathcal{N}$ is the R-symmetry index and $a, \dot{a} = 1, 2$ are respectively spinor indices and

$$\bar{Q}_{I\dot{a}} = (Q^I_a)^\dagger\ .$$

- The super-Poincaré algebra is

$$\begin{aligned}
\{Q^I_a, Q^J_b\} &= \epsilon_{ab} Z^{IJ}\,, &\qquad \{\bar{Q}_{I\dot{a}}, \bar{Q}_{J\dot{b}}\} &= \epsilon_{\dot{a}\dot{b}} Z^{IJ}\,,\\
\{Q^I_a, \bar{Q}_{J\dot{b}}\} &= 2i\delta^I_J (\sigma^\mu)_{a\dot{b}} P_\mu\,, &\qquad [P^\mu, Q^I_a] &= 0\,,\\
[L^{\mu\nu}, Q^I_a] &= (\sigma^{\mu\nu})_{ab} Q^I_b\,, &\qquad [L^{\mu\nu}, \bar{Q}_{I\dot{a}}] &= (\bar{\sigma}^{\mu\nu})_{\dot{a}\dot{b}} \bar{Q}_{I\dot{b}}\,,\\
[R^I{}_J, Q^K_a] &= \Gamma^K{}_J Q^I_a\,, &\qquad [R_I{}^J, \bar{Q}_{K\dot{a}}] &= -\Gamma_K{}^J \bar{Q}_{I\dot{a}}\,,\\
[R^I{}_J, L_{\mu\nu}] &= 0\,, &\qquad [R^I{}_J, P_\mu] &= 0\,,
\end{aligned} \tag{5.9}$$

  where

  - $Z_{IJ} = (Z^{IJ})^\dagger$ are the central extensions of the algebra, i.e.

    $$[Z_{IJ}, \mathit{Anything}] = 0$$

    where "Anything" includes all Poincaré and spinorial generator;
  - R^I_J are the R-symmetry group generators and $(R^I{}_J)^\dagger = R_I{}^J$ R-symmetry is an *internal* symmetry acting on I, J indices and is $U(\mathcal{N})$ for 4d super-Poincaré , i.e. R^I_J satisfy $U(\mathcal{N})$ algebra commutation relations. Q^I_a besides being a spacetime spinor (denoted by a index) is also a $U(\mathcal{N})$ spinor denoted by I index.
  - As as indicated by equations in the last line of (5.9), $\Gamma^I{}_J$ furnish an (adjoint) representation of the R-symmetry group.

▶▶ **Exercise 5.10:** *As indicated by the indices, central extensions Z_{IJ} should be in tensor representation of the R-symmetry group. Find $[R^I_J, Z_{KL}]$.*

- Number of real supercharges in 4d super-Poincaré is $4\mathcal{N}$. So 4d $\mathcal{N}$-super-Poincaré has $4\mathcal{N}$ spinor and 10 Poincaré , $\mathcal{N}^2$ R-symmetry generators. Moreover, there are possibly central elements.

- In SUSY gauge theories the $U(1)$ part of $U(\mathcal{N})$ R-symmetry is generically anomalous and hence the exact R-symmetry group is $SU(\mathcal{N})$. For $\mathcal{N} = 1$ case, the $U(1)$ R-symmetry reduces to a discrete $\mathbb{Z}_2$ subgroup, called R-parity.

- One may construct unitary irreps of super-Poincaré for any $\mathcal{N}$. Let us here focus on massless multiplets. If the lowest spin in a given massless multiplet is s, the highest spin in the same supermultiplet *can be* $s + \mathcal{N}/2$. (*Intuition*: supercharges carry spin 1/2 and states in the multiplet are constructed by action of supercharges on the lowest spin state, until we hit zero. Due to antisymmetry of supercharges this will happen with the maximum $\mathcal{N}/2$ hits.)

- Conversely, in each multiplet there is a state which is killed by the action of half (increasing-ladder operators) of supercharges. This state has the highest spin in the multiplet and one can hence use this to name the multiplet. The highest spin state is a chiral/BPS state.

- Therefore, the difference between the smallest and largest spins in a multiplet is less than or equal to $\mathcal{N}/2$.

- $\mathcal{N}=1$ massless multiplets are composed of states with spins s and $s-1/2$.

- For the same reason as above, for $\mathcal{N}>4$ we do not have a supersymmetric gauge theory. Moreover, $\mathcal{N}=8$ is the largest $\mathcal{N}$ one can have in 4d, if we want to avoid multiplets with spins higher than 2.

- In the absence of central extensions Z_{IJ}, 1/2 BPS states are all massless. Massive 1/2 BPS states can happen when central charge is non-zero; these are states with non-zero central charge of Z_{IJ} (not to be confused with central charge of a CFT.)

5.3.2 The 4d superconformal algebra commutation relations

- Superconformal algebras are in fact supersymmetric versions conformal algebra in any dimension.

- The super-algebra obtained from closure of super-Poincaré algebra after addition of dilation and special conformal transformations is nothing but the superconformal algebra.

- This will happen if for $\mathcal{N}$-super-Poincaré we add $4\mathcal{N}$ superconformal supercharges, usually denoted by $S_{Ia},\ \bar{S}^I_{\dot{a}}=(S_{Ia})^\dagger$ (of course together with the bosonic generators of the conformal group). The new (anti)commutation relations of the superconformal algebra are hence

$$\begin{aligned}
\{S_{Ia},S_{Jb}\}=\epsilon_{ab}\tilde{Z}_{IJ}\,,\qquad &\{\bar{S}^I_{\dot{a}},\bar{S}^J_{\dot{b}}\}=\epsilon_{\dot{a}\dot{b}}(\tilde{Z}^\dagger)^{IJ}\,,\\
\{S_{Ia},\bar{S}^J_{\dot{b}}\}&=-2i(\sigma^\mu)_{a\dot{b}}K_\mu\,,\\
\{Q^I_a,S_{Jb}\}=2\delta^I_J\epsilon_{ab}\,D&+\delta^I_J(i\sigma^{\mu\nu})_{ab}L_{\mu\nu}+\epsilon_{ab}R^I{}_J\,,\\
\{\bar{Q}_{I\dot{a}},S_{Jb}\}=0\,,\qquad &\{Q^I_a,\bar{S}^J_{\dot{b}}\}=0\,,
\end{aligned}\tag{5.10}$$

$$\begin{aligned}
[D,Q^I_a]=-\frac{1}{2}Q^I_a\,,\qquad &[D,S_{Ia}]=+\frac{1}{2}S_{Ia}\,,\\
[K^\mu,Q^I_a]=(\sigma^\mu)_{a\dot{b}}\bar{S}^I_{\dot{b}}\,,\qquad &[K^\mu,\bar{Q}_{I\dot{a}}]=(\bar{\sigma}^\mu)_{\dot{a}b}\bar{S}_{I\dot{b}}\,,\\
[P^\mu,S_{Ia}]=(\sigma^\mu)_{a\dot{b}}Q_{I\dot{b}}\,,\qquad &[P^\mu,S^I_{\dot{a}}]=(\bar{\sigma}^\mu)_{\dot{a}b}Q^I_b\,,\\
[L^{\mu\nu},S_{Ia}]=(\sigma^{\mu\nu})_{ab}S_{Ib}\,,\qquad &[L^{\mu\nu},\bar{S}^I_{\dot{a}}]=(\bar{\sigma}^{\mu\nu})_{\dot{a}\dot{b}}\bar{S}^I_{\dot{b}}\,,\\
[R_I{}^J,S_{Ka}]=\Gamma_K{}^JS_{Ja}\,,\qquad &[R^I{}_J,\bar{S}^K_{\dot{a}}]=-\Gamma^K{}_J\bar{S}^I_{\dot{a}}\,,\\
[R^I{}_J,K_\mu]=0\,,\qquad &[R^I{}_J,D]=0\,,
\end{aligned}\tag{5.11}$$

- Therefore, the bosonic part of the superconformal algebra is $so(4,2)\times u(\mathcal{N})\simeq su(2,2)\times u(\mathcal{N})$ and there are $2\cdot 4\mathcal{N}$ fermionic generators. 4d superconformal algebras are hence usually denoted by $su(2,2|\mathcal{N})$. These are algebras with bosonic part $su(2,2)\times G_R$ (which G_R is $u(1)$ for $\mathcal{N}=1$, $su(2)\times u(1)$ for $\mathcal{N}=2$ and, $su(4)$ for $\mathcal{N}=4$. The fermionic generators are in spinor representation of the conformal group $SU(2,2)$ as well as the spinor representation of the R-symmetry group G_R.

5.3.3 Some comments on representations of $SU(2,2|\mathcal{N})$

- The representations here can be constructed using a combination of discussions we had for conformal and super-Poincaré groups.
- Note that, as is seen from (5.11), action of Q's on a state increases the conformal weight by 1/2, while action of S's decreases that by 1/2.
- Each superconformal multiplet is hence built upon primary states (which have the lowest conformal weight in the multiplet). These are states killed by the action of K_μ.
- As in supersymmetry multiplets, if there are states which are killed by some of super-generators, we then have BPS states. If the number of real supercharges killing a state is q, the state is called $q/(8\mathcal{N})$ BPS.
- So, there could be states which are primary and BPS. The superconformal multiplets which are built upon these states are called *short* or *atypical* states/multiplets.
- Conformal multiplets based on non-BPS primary states are called *long* or *typical* multiplets.
- The vacuum state is the only state which preserves all the possible supersymmetries, it is killed by both Q's and S's.
- The next possible BPS multiplet is a half-BPS *chiral primary* multiplet, a multiplet which is based on supersymemtric chiral and conformal primary state.
- The matter content of $\mathcal{N}$-superconformal gauge theories discussed in the previous section also form a short multiplet (in fact a chiral primary multiplet) of the corresponding superconformal algebra.
- While 1/2-BPS chiral-primary states are necessarily spacetime scalars, they can carry non-trivial representation of the R-symmetry group. The conformal weight of these states are hence completely determined by their R-symmetry representation, which can be encoded by the *R-charge* of the state.
- The scaling dimension of chiral primary states, and hence all the states in the same multiplet (by the virtue of superconformal algebra) is protected, i.e. the scaling dimension of chiral primary states does not receive any perturbative (and in fact non-perturbative) corrections.

5.4 The central charges, trace anomaly and 4d c-theorem

- As we discussed in section 4, the 2d conformal algebra admits extension by central charge. Although the 2d Virasoro generators have the geometric interpretation as generators of conformal transformations (all meromorphic functions), the central charges $\mathbf{c}, \bar{\mathbf{c}}$ do not have geometric interpretations; they are properties of the conformal field theory.

- In 2d we gave two different ways to specify the central charge for a given theory:

  I. Using OPE of two energy-momentum tensors:

$$z^4\langle T(z)T(0)\rangle = \mathbf{c}/2\,, \qquad \bar{z}^4\langle \bar{T}(\bar{z})\bar{T}(0)\rangle = \bar{\mathbf{c}}/2.$$

  II. Using the expression for the trace anomaly (4.156):

$$\langle T_{\bar{z}z}\rangle = -\frac{1}{12}\mathbf{c}\mathcal{R}\,, \qquad \langle T_{z\bar{z}}\rangle = -\frac{1}{12}\bar{\mathbf{c}}\mathcal{R}\,,$$

  where $\mathcal{R}$ is the Ricci curvature of the 2d surface the CFT is defined on.

- One may extend the above two notions of the central charge to higher dimensions.
- In higher dimensions, though, the central charge will not appear as the central extension of the corresponding conformal algebra $so(d,2)$.

5.4.1 Stress tensor central charge $\mathbf{c_T}$

- **Stress tensor central charge $\mathbf{c_T}$**, for a d dimensional CFT is defined as,

$$\langle T_{\mu\nu}(x)T_{\alpha\beta}(0)\rangle = \frac{\frac{\mathbf{c_T}}{2}}{S_d^2}\frac{1}{|x|^{2d}}M_{\mu\nu\alpha\beta} \tag{5.12}$$

where $S_d = 2\pi^{d/2}\Gamma(d/2)$ is the volume of unit radius S^{d-1} and we assumed $\langle T_{\mu\nu}\rangle = 0$. **NOTE:** *With the above definition $\mathbf{c}_T = \mathbf{c}$ in the 2d case, once we recall that in the 2d case $S_2 = 2\pi$ and that we "redefined" the energy momentum tensor from the canonical one $T_{\mu\nu} = -\frac{2}{\sqrt{\det g}}\frac{\delta S}{\delta g^{\mu\nu}}$, by a factor of 2π (*cf. *the comment below* (4.23)*).*

▶▶ **Exercise 5.11:** *Argue that $M_{\mu\nu\alpha\beta}$ is of the form*

$$M_{\mu\nu\alpha\beta} = I_{\mu\alpha}I_{\nu\beta} + I_{\mu\beta}I_{\nu\alpha} - \frac{2}{d}\eta_{\mu\nu}\eta_{\alpha\beta}\,, \qquad I_{\mu\nu} = \eta_{\mu\nu} - \frac{2x_\mu x_\nu}{x^2}. \tag{5.13}$$

▶▶ **Exercise 5.12:** *Show that for tensor $I_{\mu\nu}(x)$ defined above,*

$$Tr(I_{\mu\nu}) = d-2\,, \qquad I^2_{\mu\nu} = \eta_{\mu\nu}\,, \qquad \det I_{\mu\nu} = -1\,,$$
$$I_{\mu\alpha}(x)I_{\alpha\beta}(x-y)I_{\beta\nu}(y) = I_{\mu\nu}(x'-y')\,, \qquad x'^\mu = \frac{x^\mu}{x^2}\,. \tag{5.14}$$

Moreover, show that

$$M_{\mu\nu\alpha\beta}M_{\alpha\beta\rho\sigma} = 2\left(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \frac{2}{d}\eta_{\mu\nu}\eta_{\alpha\beta}\right)\,. \tag{5.15}$$

For more properties of $I_{\mu\nu}$ tensor see [`arXiv:hep-th/9307010`].

- For *free CFT* with N_s real scalars, N_f Weyl fermions, and N_v gauge fields, in four dimensions [H. Osborn, A. Petkou, Annals Phys. 231, 311 (1994), hep-th/9307010]

$$\mathbf{c}_T = \frac{4}{3}N_s + 4N_f + 16N_v \tag{5.16}$$

▶▶ **Exercise 5.13:** *Verify the above. To do so, start with a free theory bosonic theory with N_s number of fields,*

$$S = -\frac{1}{2}\int d^dx\ (\partial\phi)^2\,, \qquad \langle\phi(x)\phi(0)\rangle = \frac{1}{(d-2)S_d}\frac{1}{|x|^{d-2}}\,. \tag{5.17}$$

I. *Write down the appropriate expression for its energy-momentum tensor, which is traceless on-shell,* i.e.

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{4(d-1)}\left((d-2)\partial_\mu\partial_\nu\phi + \eta_{\mu\nu}\Box\right)\phi^2 \tag{5.18}$$

II. *Compute the VEV of two energy-momentum tensors using* (5.17).
III. *For massless d-dimensional **Dirac** fermions, first show that the appropriate energy momentum tensor is*

$$T_{\mu\nu} = \bar\psi(\gamma_\mu\overleftrightarrow{\partial}_\nu + \gamma_\nu\overleftrightarrow{\partial}_\mu)\psi\,, \qquad \langle\psi(x)\psi(0)\rangle = \frac{1}{S_d}\frac{\gamma\cdot x}{|x|^d}\,. \tag{5.19}$$

and then compute $\mathbf{c}_T$.
IV. *Only in four dimensions, free vector gauge fields also have a traceless stress tensor and form free CFT. Compute contribution of gauge fields to $\mathbf{c}_T$ starting from their energy momentum tensor*

$$T_{\mu\nu} = F_{\mu\alpha}F^\alpha{}_\nu - \frac{1}{4}F^2\eta_{\mu\nu}\,, \tag{5.20}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the gauge field strength.

5.4.2 Trace anomaly central charges, a, c

- In 4d, unlike 2d case, there are two geometric curvature invariants of scaling dimension four. These are combinations which can appear in the RHS of the trace anomaly equation:

$$\langle T^\mu_\mu\rangle = -\frac{\mathbf{c}}{(8\pi)^4}(\text{Weyl})^2 + \frac{\mathbf{a}}{(8\pi)^4}(\text{Euler})\,, \tag{5.21}$$

where

$$\begin{aligned}(\text{Weyl})^2 &= C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta} = R^2_{\mu\nu\alpha\beta} - 2R^2_{\mu\nu} + \frac{1}{3}R^2\,,\\ \text{Euler} &= \frac{1}{4}\epsilon^{\mu\nu\alpha\beta}\epsilon^{\rho\sigma\delta\gamma}R_{\mu\nu\rho\sigma}R_{\alpha\beta\delta\gamma} = R^2_{\mu\nu\alpha\beta} - 4R^2_{\mu\nu} + R^2\end{aligned} \tag{5.22}$$

and $C_{\mu\nu\alpha\beta}$ is the Weyl curvature tensor. Note that "Euler" is basically the same as Gauss-Bonnet term. The above curvature invariants are computed for the 4d spacetime the CFT is defined on.

- We remark that, in a supersymmetric theory with external source $V_\mu(x)$ for the R-current, there is an additional term to the RHS of (5.21) of the form $\frac{\mathbf{c}}{6\pi^2}V_{\mu\nu}^2$, where $V_{\mu\nu}$ is field strength of V_μ.
- For *free CFT*'s with N_s real scalars, N_f Weyl fermions and N_v vector fields [`N. D. Birrell, P. C. W. Davies, Quantum Fields in Curved Space, Cambridge, Univ. Press (1982)`]:

$$\mathbf{c} = \frac{1}{10}\left(\frac{4}{3}N_s + 4N_f + 16N_v\right) , \qquad \mathbf{a} = \frac{1}{45}\left(2N_s + 11N_f + 124N_v\right) . \tag{5.23}$$

- As we see $\mathbf{c}_T \propto \mathbf{c}$, precisely $\mathbf{c}_T = 10\mathbf{c}$.
- One can show that $\mathbf{a}$ is related to four point function of four energy momentum tensors.

▶▶ **Exercise 5.14:** *Using* (5.23) *for an* $\mathcal{N}=1$ ***free theory*** *with* N_v *vector multiplets and* N_f *chiral multiplets show that*

$$\mathbf{c} = \frac{2}{3}(3N_v + N_f) , \qquad \mathbf{a} = \frac{1}{5}(9N_v + N_f) . \tag{5.24}$$

▶▶ **Exercise 5.15:** *For* $\mathcal{N}=4$ *SCFT where we have a fixed line, the expressions of* $\mathbf{c}$ *and* $\mathbf{a}$ *are expected to be exact and independent of the coupling. Using* (5.24)*, show that for an* $SU(N_c)$ $\mathcal{N}=4$ *SYM theory*

$$\mathbf{c} = \frac{5}{3}\mathbf{a} , \qquad \mathbf{a} = \frac{12}{5}(N_c^2 - 1) . \tag{5.25}$$

With a similar reasoning, for $\mathcal{N}=2$ $SU(N_c)$ *with* $N_{hyper} = 2N_c$ *(with hypers in the fundamental rep), where the theory has a fixed conformal line, show that*

$$\mathbf{a} = \frac{2}{5}(7N_c^2 - 5) , \qquad \mathbf{c} = \frac{8}{3}(2N_c^2 - 1) . \tag{5.26}$$

- As pointed out earlier, the $U(1)$ part of $U(\mathcal{N})$ R-symmetry of 4d SCFT's is anomalous. Using superconformal algebra it has been shown that the anomaly of $U(1)_R$ is related to the trace anomaly coefficients $\mathbf{a}, \mathbf{c}$. For example for $\mathcal{N}=1$ theories

$$\mathbf{a} = \frac{3}{32}\left(3\mathrm{Tr}\,(R_{\mathcal{N}=1}^3) - \mathrm{Tr}\,(R_{\mathcal{N}=1})\right) , \qquad \mathbf{c} = \frac{1}{32}\left(9\mathrm{Tr}\,(R_{\mathcal{N}=1}^3) - 5\mathrm{Tr}\,(R_{\mathcal{N}=1})\right) , \tag{5.27}$$

where the trace is over all species of Weyl fermions, and $R_{\mathcal{N}=1}$ is the generator of $U(1)_R$ symmetry in the $\mathcal{N}=1$ superconformal algebra. For more details and references e.g. see

`D. Anselmi, D.Z. Freedman, M.T. Grisaru, A.A. Johansen, Nucl. Phys. B 526 (1998) 543, hep-th/9708042,`
`D. Anselmi, J. Erlich, D.Z. Freedman, A.A. Johansen, Phys. Rev. D 57 (1998) 7570, hep-th/9711035,`
`A.D. Shapere, Y. Tachikawa, JHEP 0809 (2008) 109, arXiv:0804.1957[hep-th].`

- This relation, among other things, implies that there is a combination of scaling and $U(1)_R$ Nöther currents which is non-anomalous and conserved.

- As we expected **c**, **a** are functions of the operator content of the CFT. It has been argued [see recent papers by `Slava Rychkov`] that the central charges satisfy a universal lower bound which is a function of the dimensions of the lowest and second- lowest scalars present in the CFT.

▶▶ **Exercise 5.16:** *In general d dimensional case work out the "Ward identity" for trace anomaly involving two energy-momentum tensors. That is, show that*

$$\begin{aligned} \langle T^\rho_\rho(x) T_{\mu\nu}(y)\rangle &= 0\,, \\ \langle T^\rho_\rho(x) T_{\mu\nu}(y) T_{\alpha\beta}(z)\rangle &= 2\left(\delta^d(x-y)+\delta^d(x-z)\right)\langle T_{\mu\nu}(y) T_{\alpha\beta}(z)\rangle \end{aligned} \tag{5.28}$$

5.4.3 Four dimensional c-theorem?!

- One can compute the central charges **c** and **a**, starting from (5.21) for any given QFT. The theory need not be conformal or at its conformal fixed point. In this case **c** and **a** are in general functions of the couplings of the theory, as well as the details of its matter content (d.o.f).

- In other words, **c** and **a** have running along the RG flow.

- Let us suppose that we start with an asymptotic free theory. Being free, in the UV the theory is conformal and it is central charges are easily computed, e.g. as given in (5.23). Denote them by $\mathbf{c}_{UV}$, $\mathbf{a}_{UV}$.

- Suppose that the theory has a conformal fixed point in the IR, with $\mathbf{c}_{IR}$, $\mathbf{a}_{IR}$. The question is whether we have something similar to Zalolochikov's **c**-theorem for the 4d theories? Is $\mathbf{c}_{UV} \geq \mathbf{c}_{IR}$ or $\mathbf{a}_{UV} \geq \mathbf{a}_{IR}$ always true for any QFT?

- We know now that *there is no 4d* **c***-theorem*, but there are strong evidence for having an **a**-theorem.

- 4d **a**-theorem was originally conjectured by J. Cardy in `J. Cardy, Phys. Lett. B215, 749 (1988).`
  See `H. Osborn, Phys. Lett. B222, 97 (1989),`
  `I. Jack, H. Osborn, Nucl. Phys. B343, 647-688 (1990).`
  for early attempts to prove it.

- There are recent works basically proving the 4d **a**-theorem. The main paper is
  `Z. Komargodski, A. Schwimmer, JHEP 1112, 099 (2011) [arXiv:1107.3987 [hep-th]].`
  See also
  `M. A. Luty, J. Polchinski, R. Rattazzi, JHEP` **1301** `(2013) 152 [arXiv:1204.5221 [hep-th]].`

6 3d CFT's

Although there has been strong evidence for existence of 3d CFT's since the early 1980's, it was not clear whether they admitted Lagrangian description. Since the mid-1990's existential evidence were mainly based on string and D-brane theory results and received further support by the AdS/CFT. These latter are beyond the scope of our lectures but has its own interesting literature. Since 2007, there has been a lot of progress in constructing explicit actions for 3d CFT's, what we will review very briefly. Let us however, first start with a specification of our earlier discussions in sections 2 and 3 to 3d case.

6.1 Review of our earlier results

- The conformal group in 2+1 dim. is $SO(3,2)$ (and in 3d is $SO(4,1)$. At the level of the algebra, and as far the spinor representations are concerned $spin(3,2) = sp(4,\mathbb{R})$.
- The representations of conformal group are hence labeled by the 3d spin and scaling dimension. The unitarity bound for this representations are (*cf.* (3.22))

$$\begin{aligned}
\textbf{Scalar}: &\qquad \Delta_0 \geq \frac{1}{2}\,,\\
\textbf{Spin 1/2}: &\qquad \Delta_0 \geq 1\,,\\
\textbf{Vector}: &\qquad \Delta_0 \geq 2\,,\\
\textbf{Antisymmetric } F_{\mu\nu}\textbf{:} &\qquad \Delta_0 \geq 3/2\,,\\
\textbf{Symmetric traceless } T_{\mu\nu}\textbf{:} &\qquad \Delta_0 \geq 3\,,
\end{aligned} \tag{6.1}$$

- The spacetime dependence of two and three point correlators of primary operators in 3d CFT's is completely fixed by the conformal invariance and four and higher point functions of primary is restricted to be only through conformal cross-ratios.
- Four point function of primaries can be reduced to a sum over correlators of three primary operators and 3d conformal blocks (analogously to discussions in section 4.9.2). Conformal blocks only depend on the spectrum of the primaries and possibly the central charge (see below for more on the latter).

6.2 Examples of 3d CFT's

- As 4d cases, 3d CFT's can appear in the non-trivial RG fixed points of usual 3d QFT's. Of course the fixed point, if it exists, may not be perturbatively accessible and hence the theory may not admit a simple Lagrangian description, even if we have various pieces of evidence that the fixed point and the CFT exists.
- So, let us examine the simplest QFT's we know of, i.e. a scalar theory e.g. in the form of an $O(N)$ vector or matrix model, and gauge theory.

- Consider e.g. a vector $O(N)$ model with $(\phi_a\phi_a)^2$ type interaction. This coupling is relevant in 3d and the theory may have a nontrivial IR fixed point. In fact this specific theory is argued to flow to a nontrivial CFT in the IR, but the CFT may not be of the form of standard QFT's. For more details and discussions see `I. Klebanov, A. M. Polyakov, hep-th/0210114` and its followups.

▶▶ **Exercise 6.1:** *Any free field, in any dimension is trivially conformal. Consider a free 3d QFT with N_s real scalar fields, N_f Majorana fermions and N_v Maxwell $U(1)$ gauge fields. Compute the stress tensor central charge for this theory defined in* (5.12). *Start with a free action for each case, write down the energy momentum tensor $T_{\mu\nu}$ and compute $\langle T_{\mu\nu}(x)T_{\alpha\beta}(y)\rangle$ two point function.*
Answer: $\mathbf{c}_T = \frac{3}{2}(N_s + N_f)$.
*See **H. Osborn, G.M. Shore, arXiv:hep-th/9909043,** for more details.*

6.2.1 (Super)conformal gauge theories

- Let's next consider the 3d gauge theory case. In 4d case the YM gauge theory is dimensionless with possibly negative β-function and hence we had the possibility of finding asymptotic free CFT's within this class of theories.
- In generic d dimensions the YM gauge coupling g^2 has mass dimension $4-d$. So, the dimensionless coupling is $\Lambda_0^{d-4}g^2$, for some energy scale/cutoff Λ_0. Therefore, for $d<4$ the theory is super-renormalizable and non-renormalizable in $d>4$.
- 3d YM theory is free in the UV (dimensionless coupling is small) and flows to strong coupling (possibly in a fixed point) in the IR. If in 3d there is a non-trivial fixed point in the IR, the YM theory flows to a possibly 3d CFT. In this way one may obtain a 3d CFT.
- As nowadays a standard practice, to have a control or protection over quantum corrections, it is convenient to add some amount of supersymmetry and consider SYM and study its RG flows. Using this idea one may also use the tools available in string or M-theory gadget. All in all, what we have learned from these is the existence of this nontrivial IR fixed point and that at this fixed point we are dealing with a 3d superconformal field theory (SCFT).
- However, the question whether this theory admits an action description was not answered until 2006, when the work of Bagger-Lambert and Gustavsson (BLG), `J.Bagger and N.Lambert, hep-th/0611108, arXiv:0711.0955, arXiv:0712.3738[hep-th]; A.Gustavsson, arXiv: 0709.1260[hep-th]` indicated existence of an $\mathcal{N}=8$ 3d QFT, which was conformal invariant.
- After that the work of ABJM, `O. Aharony, O. Bergman, D. Jaferis, J. Maldacena, arXiv:0806.1218 [hep-th]`, put this into a more standard supersymmetric Chern-Simons gauge theory language:

- The idea of using super-CS theory to model 3d CFT was initiated by `J. Schwarz, hep-th/0411077`, and after `M. Van Raamsdonk, hep-th/0803.3803` showed the BLG theory can be written in terms of $SU(2) \times SU(2)$ Super-CS theory.
  - ABJM theory is an $SU(N) \times SU(N)$ $\mathcal{N} = 6$ Chern-Simons gauge theory.
  - Despite of being a CS theory, the theory is **parity even**, because the parity just exchanges the two gauge groups.
  - ABJM theory has hence two parameters: level of the CS, which is an integer k, and rank of the gauge group N. The large N behavior of the theory is governed by N/k which plays the role of 't Hooft coupling.
  - For generic values of k the theory has $\mathcal{N} = 6$ supersymmetries, i.e. 12 real-valued 3d Poincaré supercharges. However, for $k = 1, 2$ the SUSY is enhanced and the theory has $\mathcal{N} = 8$, i.e. 16 super-Poincaré charges.
  - The theory is conformal and has hence the same number of super-conformal supercharges, i.e. it has altogether 24 supercharges for generic k and 32 for $k = 1, 2$.

- Since then there has been many many followup papers exploring various extensions of the ABJM idea to cases with less SUSY and more varieties of matter fields. Now, we seem to have a classification of 3d superconformal Chern-Simons field theories with $\mathcal{N} \geq 4$. For a review and collection of references e.g. see `X. Chu, Master thesis at http://fy.chalmers.se/ tfebn/YongsMScthesis.pdf`.

- The BLG and ABJM models and the followups are related to the problem of low energy effective action for multiple M2 branes in M-theory. This topic is beyond the scope of our lectures, however, interested reader is encouraged to look at `J. Bagger, N. Lambert, S. Mukhi, C. Papageorgakis, arXiv:1203.3546[hep-th]`.

6.3 3d Superconformal Algebras $OSp(\mathcal{N}|4)$

- The 3d conformal algebra is $so(3, 2) \simeq sp(4, \mathbb{R})$.

- One may extend it to suerconformal algebra by addition of supercharges and R-symmetry generators.

- 3d spinors are constructed from two Weyl 2d fermions. As discussed in section 4, in 2d spinors can be simultaneously Majorana and Weyl. Therefore, the 3d fermions are two component *real-valued*, say denoted by $Q_a,\ a = 1, 2$.

- $\mathcal{N}$-extended 3d super-Poincaré hence involves Q^I_a, $I = 1, \cdots \mathcal{N}$ and has $2\mathcal{N}$ supercharges.

- $\mathcal{N}$-extended 3d superconformal algebra besides Q^I_a has $2\mathcal{N}$ superconformal supercharges S^I_a.

- Dealing with real-valued spinors, the R-symmetry group, acting on I index of the supercharges, is $SO(\mathcal{N})$.

- Due to Kac classification of superalgebras
  `V. Kac, Comm. Math. Phys., 53: 31, (1977),`
  the $\mathcal{N}$-extended 3d superconformal algebra is hence $OSp(\mathcal{N}|4,\mathbb{R})$. This is a superalgebra
  - whose bosonic part is $SO(\mathcal{N})_R \times Sp(4,\mathbb{R})_{conf}$,
    ▶▶ **Exercise 6.2:** *How many bosonic generators are there?*
  - its $4\mathcal{N}$ fermionic generators are spinors $\mathcal{N}$-vectors of R-symmetry group while a four component $sp(4)$ real-valued spinors. That is, they are of the form $\mathcal{Q}^I_\alpha$ where $I = 1, \cdot, \mathcal{N}$ and $\alpha = 1,2,3,4$. $\mathcal{Q}^I_\alpha$ may be decomposed into $Q^I_a,\ S^I_a$.
  - The full commutations relations of $Osp(\mathcal{N}|4)$ are:

$$\begin{aligned}\{\mathcal{Q}^I_\alpha, \mathcal{Q}^J_\beta\} &= \delta^{IJ}(i\Gamma^{\mu\nu})_{\alpha\beta}\mathcal{M}_{\mu\nu} + S_{\alpha\beta}(i\Gamma^{IJ})\mathcal{R}_{IJ}\,,\\ [\mathcal{M}_{\mu\nu}, \mathcal{Q}^I_\alpha] &= (i\Gamma^{\mu\nu})_{\alpha\beta}\mathcal{Q}^I_\beta\,,\\ [\mathcal{R}_{IJ}, \mathcal{Q}^K_\alpha] &= (i\Gamma^{IK})\mathcal{Q}^J_\alpha - (i\Gamma^{JK})\mathcal{Q}^I_\alpha\,,\end{aligned} \tag{6.2}$$

    where $\mathcal{M}_{\mu\nu},\ \mu,\nu = 1,2,3,4,5$ are the $so(3,2)$ generators and $\mathcal{R}_{IJ},\ I,J = 1,\cdots,\mathcal{N}$ are $SO(\mathcal{N})$ R-symmetry generators. $S_{\alpha\beta}$ is the symplectic metric on $sp(4)$ and Γ's are corresponding Dirac Γ-matrices of the appropriate group, $SO(3,2)$ or $SO(\mathcal{N})$.
  - For more material on 3d superconformal algebras see
    `Jeong-Hyuck Park, http://arxiv.org/pdf/hep-th/9910199.pdf.`
- **A bit more on matter content of ABJM theory:**
  - As mentioned ABJM theory is a $SU(N) \times SU(N)$ CS gauge theory with $\mathcal{N} = 6$ supersymmetries. Therefore,
  - the theory has a two $SU(N)$ gauge fields. Nonetheless, being governed by CS action, there are not propagating d.o.f.
  - The theory has 4 fermions in bi-fundamental representation of the gauge group. That is, they are in $(N,\bar{N})\bigoplus(\bar{N},N)$ representation. The spinors may be arranged such that they are in spinor rep. of the R-symmetry algebra $so(6) \simeq su(4)$.
  - There are similar number of complex scalars. That is, we have four complex scalars in **4** of $su(4)_R$ while in bifundamental $(N,\bar{N})$ of gauge group.
  - SUSY implies that there are potential terms of the $(scalar)^6$ and $(scalar)^2(fermion)^2$ terms in the action.
    ▶▶ **Exercise 6.3:** *Construct all Lorentz and gauge invariant marginal operators which could be made from the above mentioned gauge, fermion and scalar fields.*

7 6d CFT's

- Perhaps the least understood and studied among CFT'd is the 6d case. In this case the conformal group is $SO(6,2) \simeq SO^*(8)$.

- There are trivial (free) field theory examples of 6d CFT's, which may involve scalar, fermionic, gauge field, as well as two-form fields.

- Six is the minimum dimension where the option of two-form field $B_{\mu\nu}$ as an independent d.o.f which cannot be rendered in terms of scalar or vector fields appears. The "field strength of B, $H = dB$, is a three form and is hence non-dynamical in 3d; it is Hodge dual to a one-form and hence dual to a pseudoscalar in 4d; it is dual to a vector in 5d and in 6d it is dual to (another) two-form.

  ►► **Exercise 7.1:** *For a free 6d CFT consisting of N_s real scalars, N_f Weyl fermions and N_v vectors and N_B two-form fields with Lagrangian $-\frac{1}{12}H^2$, compute the stress tensor central charge $\mathbf{c}_T$ defined in* (5.12).
  Answer: $\mathbf{c}_T = \frac{6}{5}N_s + 12N_f + 2\cdot 54N_B$. *If the form field is self-dual then the contribution is halved.*
  See `hep-th/9911135`.

  NOTE: *In any dimension other than two, a generally covariant gravity theory cannot be well-defined conformal field theory. This is due to the fact that two derivative theory of gravity (Einstein Hilbert) involves a dimensionful parameter the Newton constant. The higher derivative theories of gravity, like Gauss-Bonnet or Weyl gravity, have the generic feature that they involve negative-norm states (ghosts).*

- The unitarity (lower) bound on scaling dimensions of the fields are given by the scaling dimension of the corresponding free fields.

7.1 6d superconformal algebras $OSp(6, 2|2\mathcal{N})$

- As in three and four dimensions, addition of supersymmetry improves the control over the running of parameters/coupling and also keeps some of physical observables and correlation functions protected.

- In 6d we have CPT invariant Weyl fermions and the SUSY algebra can only involve one chirality. The minimal SUSY in 6d, $\mathcal{N} = 1$, hence only involves one $so(5, 1) \sim su(4)$ fermions, which has four complex components. We can have $(\mathcal{N}_L, \mathcal{N}_R)$ super-Poincaré which have $\mathcal{N}_L$ left handed and $\mathcal{N}_R$ right handed supercharges, altogether $8(\mathcal{N}_L + \mathcal{N}_R)$ real supercharges.

- Nahm's classification of Superconformal algebras (`W. Nahm, Nucl. Phys. B135 (1978) 149,`) tells us that there the maximum possible dimension of having superconformal algebra is $d = 6$. In higher dimensions we cannot have conformal (field) theories. In a similar way, SUSY restricts the largest spacetime dimension we can have supergravity to be eleven.

- Moreover, according to Nahm's classification only 6d chiral super-Poincaré $(0, \mathcal{N})$ admit a conformal extension.

- The two possible 6d superconformal algebras associated with $(0,1)$ and $(0,2)$ super-Poincaré are $Osp(6,2|2\mathcal{N}) \simeq Osp(6,2|2\mathcal{N})$ superconformal algebras (with $\mathcal{N} = 1, 2$, respectively). For higher $\mathcal{N}$ the smallest rep. will involve spins larger or equal to two and are hence not physically associated with a CFT.

- The bosonic part of $osp(6,2|\mathcal{N})$ superalgebra is $so(6,2)_{conf} \times sp(2\mathcal{N})_R$. (Note that $sp(2) \simeq su(2)$ and $sp(4) \simeq so(5)$.)
  ▶▶ **Exercise 7.2:** *What is the dimension of the bosonic part of* $OSp(6,2|2\mathcal{N})$ *super-group?*

- The supercharges $\mathcal{Q}_{A\alpha}$, where $A = 1, \cdots, 8$ is the $so^*(8)$ Majorana-Weyl index and $\alpha = 1, 2$ is the $su(2)$ (for $\mathcal{N} = 1$) or $\alpha = 1, 2, 3, 4$ is the $sp(4)$ spinor index (for $\mathcal{N} = 2$). We note that superconformal charges $\mathcal{Q}_{A\alpha}$ may be decomposed as $(Q_{I\alpha}, \bar{S}^I_{\dot\alpha})$ where $Q_{I\alpha}$ are 6d super-Poincaré $SO(5,1)$ Weyl spinors and $\bar{S}^I_{\dot\alpha}$ are 6d superconformal supercharges. For $\mathcal{N} = 1$, $Q^\dagger_{I\alpha} = \bar{Q}^I_{\dot\alpha}$ (and similarly for superconformal charges $\bar{S}^I_{\dot\alpha}$, i.e. $so(5,1)$ and $su(2)$ chiralities are linked to each other and hence for $\mathcal{N} = 1$, $OSp(6,2|2)$ has altogether 16 real supercharges. With a similar argument $\mathcal{N} = 2$ superalgebra has 32 real supercharges.

- There are hence $16\mathcal{N}$ real supercharges.

$$\{\mathcal{Q}_{A\alpha}, \mathcal{Q}_{B\beta}\} = 2(i\Gamma^{IJ})_{AB}\mathcal{M}_{IJ}\delta_{\alpha\beta} + 2(i\Gamma^{ij})_{\alpha\beta}\mathcal{R}_{ij}\delta_{AB} + \mathcal{Z}_{[AB],[\alpha\beta]}\,, \tag{7.1}$$

  where Γ^{IJ} and Γ^{ij} are respectively 8×8 $so^*(8)$ and $2\mathcal{N}\times 2\mathcal{N}$ $sp(2\mathcal{N})$ Dirac matrix commutators; $\mathcal{M}_{IJ}$ is $so(6,2)$ generators and $\mathcal{R}_{ij}$ are $sp(2\mathcal{N})$ generators. $\mathcal{Z}$ term which has both $so^*(8)$ and $sp(2\mathcal{N})$ antisymmetric indices, denotes all possible central extensions.

  NOTE: *On the LHF of* (7.1)*, we have an object with* $16\mathcal{N} \times 16\mathcal{N}$ *symmetric matrix indices. On the RHS, however, we have decomposed it into* $\mathcal{M}_{IJ}$ *(there are 28 of them) and* $\mathcal{R}_{ij}$ *there are* $\mathcal{N}(2\mathcal{N}+1)$ *of them. Some of the rest can hence be accommodated in the central charges.*

  ▶▶ **Exercise 7.3:** *How many* $\mathcal{Z}$*'s are there? The indices on* $\mathcal{Z}$ *are spinorial indices. One may decompose them in terms* $so(6,2)$ *and* $sp(2\mathcal{N})$ *tensors (or forms) using* $\Gamma^{IJ\cdots K}$ *and* $\Gamma_{ij\cdots k}$ *matrices. What are the these forms?*

- The smallest reps of $\mathcal{N} = (0,1)$ in 6d are chiral multiplet with four real scalars and a Weyl fermion and the gauge multiplet containing a gauge field and a Weyl fermion.

- The smallest $\mathcal{N} = (1,1)$ multiplet is obtained from putting together a $\mathcal{N} = (0,1)$ gauge multiplet and a $\mathcal{N} = (1,0)$ chiral multiplet. The 6d gauge multiplet hence contains a gauge field, four real scalars and two opposite chirality Weyl fermions. The smallest $\mathcal{N} = (0,2)$ multiplet, on the other hand, contains a self-dual two-form, five real scalars and two Weyl fermions of similar chirality.

  See `S. Ferrara, E. Sokatchev, hep-th/0001178; J-H. Park, hep-th/9807186` for representations of 6d superconformal algebras.

7.2 The $(0, 2)$ theory as a candidate for 6d CFT

- Now we would like to discuss if there is indeed any *local* QFT which exhibits 6d conformal or superconformal invariance?

- Let's start with YM gauge theory as a possible candidate. Yang Mills coupling in $d > 4$ has negative mass dimension and hence YM gauge theory in $d > 4$ is not asymptotic free and flows to a strongly coupled theory.

- The UV completion of 5d and 6d YM theories is believed (based on evidence coming from string, brane and M-theory) to flow to a non-trivial interacting *local* 6d CFT. This CFT is expected not to involve light vector (gauge) field d.o.f.

- The $(0, 2)$ super-Poincaré admits a central extension which corresponds to a self-dual two-form field. That is a two-from B such that ${}^*dB = dB$. States which carry this central charge are one-dimensional objects, strings. These states are hence 1/2-BPS states.

  ▶▶ **Exercise 7.4:** *Show the above. Start from 6d super-Poincaré and analyze possible central extensions.*

  ▶▶ **Exercise 7.5:** *Count number of propagating d.o.f of the 6d self-dual two-form field and show that it is three.*

  ▶▶ **Exercise 7.6:** *Compute the stress tensor central charge* $\mathbf{c}_T$ *for a free* $(0, 1)$ *theory containing* N *6d chiral multiplets.* ***Answer:*** $\mathbf{c}_T = \frac{84}{5} N$.
  Compute the stress tensor central charge $\mathbf{c}_T$ *for a free* $(0, 2)$ *theory containing* N *of its smallest multiplet.* ***Answer:*** $\mathbf{c}_T = 84N$.

- It is expected that the 6d CFT which has $(0, 2)$ SUSY should involve this self-dual two-form. This self-dual two-form is part of the supermultiplet which also contain five real scalars. This theory is also called "little string theory".

- We only know that the $(0, 2)$ theory exists. We know that it arises as UV completion of 5d SYM theory. We have a matrix model formulation for its DLCQ description. However, we do not have an explicit action for it. This is an active field of research to construct this. As some recent attempts, see recent paper by `N. Lambert`.

7.3 Trace anomaly and a-theorem in 6d

- **Trace anomaly in generic even dimension** d

$$\langle T^\mu_\mu \rangle = \sum_i \mathbf{c}_i \mathcal{I}_i - (-1)^{d/2} \mathbf{a} \text{ Euler}_d \tag{7.2}$$

where $\mathcal{I}_i$ are combinations of Riemann curvature to power $d/2$ and Euler_d is the topological term Euler density Tr $(\mathcal{R} \wedge \cdots \wedge \mathcal{R})$, where $\mathcal{R}$ is the curvature two form. $\mathbf{c}_i, \mathbf{a}$ are the central charges of the theory. One of them is the stress tensor central charge $\mathbf{c}_T$.

▶▶ **Exercise 7.7:** *Work out all possible curvature combinations which may arise in the 6d trace anomaly. Refs.*
`L. Bonora, P. Pasti and M. Bregola, Class. Quant. Grav. 3 (1986) 635,`
`S. Deser, A. Schwimmer, Phys. Lett. B 309, 279 (1993) [arXiv:hep-th/9302047].`

- See `http://arxiv.org/abs/arXiv:1205.3994` for preliminary discussions on a 6d **a**-theorem.

A Spinor representations in various dimensions

- For the spinor reps of $SO(d)$ the starting point is to construct γ matrices. These are solutions to the d dimensional Clifford algebra

$$\{\gamma^a, \gamma^b\} = 2\delta^{ab}\,, \qquad a,b = 1,2,\cdots,d \tag{A.1}$$

- As a role of thumb the solutions to above are:
  - The rank of $SO(d)$ is $n = [d/2]$, which is $d/2$ in even dimensions and $(d-1)/2$ in odd dimensions.
  - In any even dimension $d = 2n$, $2^n \times 2^n$ matrices for 2^n component **Dirac** fermions.
  - In even dimensions one can show that we have "Γ" matrix,

$$\Gamma = \gamma^1\gamma^2\cdots\gamma^d \tag{A.2}$$

  which commutes with the generators of the $SO(d)$ rotation group.
  ▶▶ **Exercise A.1:** *Show the above noting that* $L_{ab} = \frac{1}{4}[\gamma_a, \gamma_b]$ *form a representation of* $SO(d)$*.*
  Using the Γ matrix one can reduce the Dirac fermions into two 2^{n-1} component **Weyl** fermions:

$$\Gamma\Psi = \pm\Psi\,. \tag{A.3}$$

  In the Weyl representation, Γ is basically equal to identity (or minus-identity).
  NOTE: *In general the fermionic representations are COMPLEX-valued.*
  - In odd dimension $d = 2n+1$, fermions (and corresponding γ matrices) are constructed from two Weyl fermions of one lower even dimension. That is, fermions are in general 2^n dimensional.

- Although in general Dirac fermions in various dimensions are complex-valued, it is possible to construct real-valued **Majorana** fermions, using the **charge conjugation** operator C:

$$C\Psi C^{-1} = \Psi\,. \tag{A.4}$$

Majorana fermions hence form 2^n component real fermions, where $n = [d/2]$.

- In general even dimension $d = 2n$ the charge conjugation operator C and the chirality operator Γ do not commute and hence cannot be diagonalized simultaneously. However, one can show that in $d = 4k + 2$ C and Γ commute and we can have **Majorana-Weyl** fermions.
  That is, in $d = 2, 6, 10, \cdots$ the irreducible fermionic representation is 2^n real-valued spinors.

- For further reading on spinor representations see the Appendix of Volume II of Plochinski's string theory book.