A Course on AdS/CFT

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This is a PhD level course, designed for second year PhD students in Theoretical High Energy Physics (HEP-TH) area and assumes a background knowledge of Quantum Field Theory at the level of Peskin-Schroder book and General Relativity. It would be great if the audience are already familiar with basics of CFT's and black hole physics. In a sense this course is a natural continuation of the black hole and CFT courses I gave at IPM since last year. I will assume no knowledge of string theory.

AdS/CFT or in more practical sense gauge/gravity duality has dominated HEP-TH area for the last 15 years. Although it appeared through developments in string theory and it is still believed that the complete formulation of it needs string theory, for most practical purposes it does not involve formulation of string theory and supersymmetry in an essential way.

AdS/CFT states that quantum gravity (string theory) on AdS_{d+1} background is described by a non-gravitational d dimensional QFT. Depending on d and the details of the matter content or low energy degrees of freedom of the QGr theory the corresponding dual QFT would be different. This QFT is expected to flow to a conformal fixed point in the UV and hence the QFT in general may be viewed as a deformation of a CFT, hence justifying the name AdS/CFT.

Being a duality one may try to use it as a handle on tackling questions about the "other side" starting from one. So far, however, it has mainly be used studying strongly coupled QFTs than the learning about QGr (from CFT knowledge). Nonetheless, in my opinion, in the coming years the QGr aspects will receive an increased attention.

Historically there were some developments prior to the stated landmark of AdS/CFT, the celebrated Maldacena's paper of 1997, which were crucial in uncovering the duality. The most notable ones are perhaps, 't Hooft's $1/N$ expansion, the seminal work of Polchinski in introducing D-branes into string theory and the Strominger-Vafa black hole microstate counting and the Matrix theory. The latter two introduced the notion of open/closed string duality which eventually led to AdS/CFT. However, as mentioned to state AdS/CFT one need not follow the historical path, having some knowledge of these is essential in understanding AdS/CFT and that is what we will partly do in this course.

Topics which will be covered here will somehow be complementary to the ones I gave in my lectures a year ago. In those lectures I mainly focused on the AdS/CFT as a natural continuation of developments started by Wilson in QFT, namely, that one can use perturb any QFT away from its RG fixed point by any local operator associated with physical observables of the theory. This gives infinitely many options to deform/perturb a QFT and the coupling of these operators are basically governed by the dual (quantum) gravity theory.

In the current course I will spend some time in reviewing the historical developments led to AdS/CFT, and focus on 't Hooft's 1/N expansion and the string theory picture based on string theory D-branes. And of course then present statement of the duality and discuss AdS/CFT as a tool to study strongly coupled QFT's and also discuss very briefly AdS/CFT as a tool to address issues about quantum gravity. These issues are still very much at research level and my discussions will mainly be introductory.

Topics to be discussed in the course

1. AdS/CFT, a historical view on its stringy perspective. This part will be about 7 sessions and will cover:
   - t' Hooft's 1/N expansion.
   - Some facts about the known CFT's in diverse dimensions.
   - A quick look into string and brane theory.
   - D-branes, the near horizon and decoupling limits, open/closed duality.
   - AdS space as solutions to (gauged) supergravities in diverse dimensions.
2. AdS/CFT, formal statement, establishing the duality. This part will be about 4 sessions and will cover:
   - Causal and geodesic structure of AdS space and some asymptotically AdS spaces.
   - Field theory on AdS space various boundary conditions.
   - Statement of the AdS/CFT duality.
   - AdS/CFT and holography.
3. AdS/CFT, a tool for strongly coupled QFTs. This part will be about 6 sessions and will cover:
   - AdS/CFT and confinement-deconfinement phase transition.
   - AdS/CFT and Wilson/Ployakov loops.
   - AdS/CFT and RG flow, holographic renormalization.
   - Holographic hydrodynamics.
4. AdS/CFT as a tool for studying quantum gravity. This part will be very brief, one session, and we will discuss general picture AdS/CFT has to offer for quantum gravity, quantum space-time and quantum aspects of black hole.

- **Texts and reading:**

  – *O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, "Large N field theories, string theory and gravity,"* Phys. Rept. **323** (2000) 183, [hep-th/9905111]..

  – *E. Witten, "Anti-de Sitter space and holography,"* Adv. Theor. Math. Phys. **2** (1998) 253 [hep-th/9802150].

  – *E. Witten, "Anti-de Sitter space, thermal phase transition, and confinement in gauge theories,"* Adv. Theor. Math. Phys. **2** (1998) 505 [hep-th/9803131].

  – *J. Polchinski, "Introduction to Gauge/Gravity Duality,"* arXiv:1010.6134 [hep-th].

  – *K. Skenderis, "Lecture notes on holographic renormalization,"* Class. Quant. Grav. **19** (2002) 5849 [hep-th/0209067].''

  – *J. Casalderrey-Solana, H. Liu, D. Mateos, K. Rajagopal and U. A. Wiedemann, "Gauge/String Duality, Hot QCD and Heavy Ion Collisions,"* arXiv:1101.0618 [hep-th].

  – *M. Rangamani, "Gravity and Hydrodynamics: Lectures on the fluid-gravity correspondence,"* Class. Quant. Grav. **26** (2009) 224003 [arXiv:0905.4352 [hep-th]].

  – *A. V. Ramallo, "Introduction to the AdS/CFT correspondence,"* arXiv:1310.4319.

- The above references contain many further references and reading material.

- This course is equivalent to *two units* and there is the possibility of formally registering for the course as a "guest student" for non-IPM students. For the latter please arrange the formal details with department office, Ms Pileroudi, niloufar@theory.ipm.ac.ir.

- The lectures will be **10-12, Sunday-Tuesday**, *in Farmanieh Bldg* and they will **start on 7th of Mehr (29 Sept. 2013)**.

Contents

1 Invitation to AdS/CFT

The AdS/CFT has dominated the last 15 years of developments in HEP-TH area. It states that quantum gravity on an AdS_{p+2} background is dual or equivalent to a $p+1$ dimensional CFT, which is a non-gravitating theory. Of course this duality is between two quantum theories, which in the CFT side is a standard quantum field theory and may be formulated through path integral. The AdS-side, the quantum gravity side, however, is not of the form of a usual quantum field theory and needs an independent definition. As we will argue below, a theory dual to a QFT in the sense of AdS/CFT cannot be a (quantum) field theory. In a different viewpoint, one may view AdS/CFT as a definition of quantum gravity (on AdS spaces).

Historically, AdS/CFT arose from string/M theory; these theories provide the framework to study quantum gravity. String/M theory was crucial in establishing and exploring AdS/CFT. In this framework, (super)gravity appears as a low energy effective theory of the quantum gravity theory. Perturbative string or M-theory is then consistently formulated on the backgrounds which are solutions to the (super)gravity theory. To have quantum fluctuations of the fields under control these backgrounds are usually required to preserve some supersymmetries. The 10d or 11d supergravities do admit (supersymmetric) solutions with AdS_{p+1} ($p \leq 6$) factors. Interestingly, the maximal supersymmetric solutions to 11d SUGRA are limited to 11d flat space, $\mathrm{AdS}_4\times\mathrm{S}^7$ and $\mathrm{AdS}_7\times\mathrm{S}^4$ (and also the $11d$ plane-wave obtained as Penrose limits over the $11d$ AdS-spacetimes); maximal SUSY solution to 10d IIA is only 10 flat space, while those of 10d IIB SUGRA is flat space and $\mathrm{AdS}_5\times\mathrm{S}^5$ (and also the $10d$ plane-wave obtained as Penrose limits over the $10d$ AdS-spacetime). There are other AdS solutions with lower number of supersymmetries and there is a very elaborate research program to classify all possible supersymmetric SUGRA solutions involving AdS factors. These solutions may be used as a starting background for AdS/CFT.

As we see generically embedding the AdS/CFT into a critical 10d superstring theory or 11d M-theory involves another "internal" space, besides the AdS factor. The exact geometry of this internal space of course depends on the details of the field content and Lagrangian (potential or superpotential) of the dual CFT. In other words, for a given CFT and its possible deformations away from the conformal fixed point, there is precisely one background geometry in the string theory or AdS side.

According to AdS/CFT, a field theory at the conformal fixed point is dual to a space with an AdS factor; the isometry of the AdS space is indeed the conformal symmetry of the dual field theory. Then, one may deform the CFT away from the fixed point by addition of the local operator with a coefficient/coupling to the action. There is hence a one-to-one correspondence between all (gauge invariant) local operators of a CFT and all of its possible deformations and also Hilbert space of the theory (recalling the operator-state correspondence).

►► **Exercise 1.1:** *Is a photon state associated with a gauge invariant operator?*

In the AdS side, any deformation of the CFT corresponds to either deforming the AdS background (usually to an asymptotic AdS geometry) or probing the AdS space with an appropriate object, a point particle, a field or an extended object like a brane. The goal of this course is to present AdS/CFT in both kinematical and dynamical levels and how the correspondences mentioned above work.

AdS/CFT duality, as evolved historically, is different than the other dualities found and studied before that; these other dualities, such as S-duality (or Seiberg duality) of (suersymmetric) gauge theories or S, T or U duality of string or M-theory were typically (not always) relating similar theories, i.e. corresponding a gauge or string theory to another gauge or string theory. AdS/CFT, however, relates a gauge field theory to a string theory. The nontrivial passage from a gauge theory to a string theory was paved by the seminal works of 't Hooft in mid 1970's and then from string theory side by the developments in string dualities in mid 1990's which led Polchinski to the discovery of D-branes and Witten's description of multiple, coincident D-branes. This is what we will review and study first in these lectures.

We will then review Maldacena's remarkable observation which followed soon afterward, that how an AdS space appears through a limit of multiple coincident D-brane geometry, and hence the birth of AdS/CFT. The key observation and idea here was actually noted and used during 1995-97 period in M(atrix)-theory: existence of *decoupling limits*, stating that

There could be corners/sectors of parameter and Hilbert spaces of a unitary theory where the dynamics is *unitarily* governed by only a subset of degrees of freedom; the physics in that corner "decouples" from the rest of the theory and is governed by a new unitary theory. If we have two different descriptions of this corner then we have a "duality" between these two descriptions.

AdS/CFT was then given a new and very fruitful description by the seminal work of Witten in early 1998, where its formulation was presented as a precise equivalence between partition functions of two quantum theories, a gauge field theory and a quantum gravity (string theory). Several pieces of evidence and further extensions and studies then followed. The AdS/CFT became the name of a new volume of "Britanica in HEP-TH" to which literally one page was added everyday. This trend still continues to date, with a lower pace though.

Witten's formulation of AdS/CFT, which will be reviewed in part two of these lectures, made it clear that classical gravity can be used to study strongly coupled quantum field theories. This led to a variant of AdS/CFT, the "gauge/gravity" correspondence where the focus on the gravity side was put on classical or semi-classical gravity processes, while still capturing inherently quantum effects in the QFT side.

One of the features of the $\text{AdS}_{d+1}/\text{CFT}_d$ or gauge/gravity correspondence is providing a very "natural" and geometric picture of the Wilsonian RG flow: the extra dimension of the gravity side, "the holographic direction", is associated with the RG scale. This, together with a direct translation of concepts and quantities of QFT into the "gravity" language, made the gauge/gravity correspondence the right framework for addressing several issues in strongly coupled field theories. More specifically,

Gauge/Gravity correspondence is a particular limit of the AdS/CFT duality where the stringy degrees of freedom in the AdS side (that is, the "genuine" quantum gravity effects) are heavy and not relevant and hence, the AdS-side can reliably be replaced with semi-classical gravity with a Lagrangian description (e.g. Einstein gravity plus possibly higher derivative corrections). The gauge theory side, however, is a full quantum, usually strongly coupled, field theory. Note that gauge/gravity correspondence *is not* obtained from AdS/CFT as a "decoupling limit". It is rather a low energy effective description of the AdS/CFT duality.

This line of research is still followed in several different aspects. These issues will be discussed in part three of these lectures.

One may recall that, after all, the precise statement of the AdS/CFT duality involves quantum gravity and hence this duality may be used as a framework to tackle various different issues and questions about quantum gravity. We will have a very brief discussion on this line of research, which is again pretty much an open area, in the last part of lectures.

1.1 Gauge/gravity correspondence, QFT viewpoint

As pointed out AdS/CFT *duality* which is an equivalence between two quantum theories, quantum gravity and a non-gravitating quantum field theory, when used as a tool for studying (strongly coupled) quantum field theories is oftentimes "reduced" to gauge/gravity *correspondence* in which the "gauge" side is a quantum field theory while the "gravity" side is a classical gravity theory (with possibly a low-lying higher derivative corrections). In this sense the gauge/gravity correspondence is reduction of AdS/CFT duality, however, from a different viewpoint it is indeed an extension of it: although the name "gauge" is there, the

QFT side need not be a gauge theory, or even to have a non-trivial conformal fixed point. The fact that any QFT has a "gravity dual" is a non-trivial statement, and if true needs discussion and establishment. This is what we would like to do in this subsection.

To this end, let us start from mid-1970's and when Wilson formulated his very remarkable Renormalization Group (RG) theory. In my opinion, that is one of the deepest developments in physics in the last 60 years which constitutes the basis for "modern field theory". In the modern QFT *any physical local operator* $\mathcal{O}_i$ may be used to deform or perturb the theory (around a given fixed point):

$$\mathcal{L} = \mathcal{L}_0 + \int \lambda_i \mathcal{O}_i(x)\,, \tag{1.1}$$

where $\mathcal{L}_0$ is the QFT Lagrangian at the fixed point. In general the "couplings" λ_i need not be constants over the spacetime, they may be $\lambda_i = \lambda_i(x)$. $\lambda_i(x)$ are different than $\mathcal{O}_i$ in the fact that in the path integral quantization of the field theory we integrate over the operators while λ_i appear a functions/numbers. The above once supplemented with unitarity of the QFT and the cluster decomposition (a manifestation of the locality of the QFT), leads to a well-defined OPE expansion and hence the deformations in (1.1) capture the most general deformation possibility.

The above makes it clear that there is a one-to-one correspondence between all the couplings and the operators. This is the basis for gauge/gravity correspondence:

$\lambda_i(x)$ are related to fields of a gravity theory and are in one-to-one correspondence with the local physical observables of the QFT.

In this viewpoint, gauge/gravity is the most natural extension of the Wilsonian RG picture and modern field theory.

But, the main question is now:

what is the theory governing the dynamics of $\lambda_i(x)$?

We will argue below that this theory, whatever it is, and regardless of the details of the quantum field theory it is "dual" to, should be

I. a gravity theory, it has a massless spin two among its d.o.f;

II. for normal QFTs with logarithmic running this is a theory in one higher dimension than the original QFT;

III. for normal QFTs with logarithmic running this is a theory on asymptotically AdS space;

IV. it has infinitely many fields, as number of λ_i (i.e. number of states in the QFT) is infinite and in principle there is no reason why λ_i should not be independent.

V. Is the gauge/gravity correspondence a full-fledged "duality", or is perhaps an approximation to a complete "duality"? If yes, can we learn from the QFT side which approximation this is; and what more can we say about the "UV completed" gravity side?

Let us go through the above five questions and points one by one.

I. We note that one of the physical observables of any QFT in any dimension, even if it does not have an explicit Lagrangian description, is its energy momentum tensor, $T_{\mu\nu}$. Energy momentum tensor, by definition, is a marginal operator, i.e. its engineering scaling dimension is dimension of the spacetime. Being a symmetric rank-two tensor, the corresponding deformation coupling should also be a similar tensor with mass dimension zero, let us call it $h_{\mu\nu}$.

On the other hand, we know that a metric in a generic $d+1$ dimensional diffeomorphism invariant theory has $(d+1)(d+2)/2-(d+1)=d(d+1)/2$ degrees of freedom.[1] These constitute the d.o.f. of a symmetric d-dimensional rank-two tensor, the same as coupling $h_{\mu\nu}$. In a more precise wording, starting from a $d+1$ dimensional metric one can always perform an ADM d plus one decomposition as

$$ds^2 = g(y)dy^2 + g_{\mu\nu}(y;x)dx^\mu dx^\nu\,. \tag{1.2}$$

In principle $g(y)$ can be positive or negative. Moreover, by a choice of "lapse" function (or $y \to \tilde{y} = f(y)$ diffeomorphism) one can eliminate g but it sign remains. So, from now on we choose, $g(y)$ to be just $\sigma = \pm 1$.

This is very suggestive that in the "gravity side" is a $d+1$ dimensional diff. invariant theory with metric among its d.o.f. Such a theory, is what is meant by a gravity theory. The fact that $T_{\mu\nu}$ is marginal means that its coupling is dimensionless, this, too, matches with specifying $h_{\mu\nu}$ with a part of the metric. In a more technical language, which will be discussed further in next lectures, the fact that $T_{\mu\nu}$ is a marginal operator corresponds to the fact that the corresponding "dual" field $h_{\mu\nu}$ is a massless field in the "gravity" side.

Some comments are in order:

- The most natural and convenient choice to realize our gauge/gravity proposal is that we identify the coordinates x^μ, i.e. the constant y surface in the metric (1.2), with the d space-time where the QFT lives.
- This argument does not fix the sign of g. That is, the *holographic direction* y can be spacelike or timelike.
- Appearance of gravity is quite general and corresponds to existence of energy momentum tensor in the QFT side.
- Diff. invariance was added by hand, it is not a direct outcome of anything in the QFT side.

[1] Of course depending on the theory usually only a part of these number of d.o.f are propagating.

II & III. As a direct outcome of the above statement of the gauge/gravity correspondence, the e.o.m for the couplings λ_i should correspond to the RG flow equation for these couplings. Explicitly, the β-function for the couplings λ_i is the e.o.m in the gravity side.

Given the above, one may now try to gain information about the metric coefficients σ and $g_{\mu\nu}(y;x)$ from the β-function equations. Let us suppose that the theory has a scalar marginal deformation, e.g. consider the $\lambda\phi^{\frac{2d}{d-2}}$ coupling in scalar field theory in d dimensions with $\mathcal{L}_0 = -\frac{1}{2}(\partial\phi)^2$. In this case we have not turned on $h_{\mu\nu}$ deformation and the coupling λ is spacetime x^μ independent and is a Lorentz scalar. Therefore, one would expect that the corresponding metric should exhibit the d dimensional Poincaré symmetry. That is, for this case the $d+1$ dimensional metric is expected to be of the form

$$ds^2 = \sigma dy^2 + F(y)^2 \eta_{\mu\nu} dx^\mu dx^\nu \,, \tag{1.3}$$

where $\eta_{\mu\nu}$ is d-dimensional Minkowski metric. Let us now focus on constant y surfaces. Moving in the y direction then corresponds to a scaling in x^μ:

$$y_1 \to y_2 \qquad \Longrightarrow \qquad x^\mu \to \frac{F(y_2)}{F(y_1)} x^\mu \,. \tag{1.4}$$

What we want to do is to reinterpret the β-function (and the sliding scales viewpoint of Wilson) as the e.o.m of a field associated with the coupling λ. To this end, we recall that the standard QFT (one-loop) computation readily leads:

$$\frac{d\lambda}{d\ln\mu} = C\lambda^\alpha \,, \tag{1.5}$$

where μ is the Wilson's sliding scale and C is a c-number and λ is a power which depends on the dimension d.

Warning: Some of the discussions above is not precise and is meant to be inspiring/illuminating. These points will hopefully become clear in the end of these lectures.

We would like to view the above as the e.o.m of a $d+1$ dimensional field theory on the space defined by metric (1.3). This equation of motion for the field configuration of our interest (λ which has no x^μ dependence) will only involve derivatives of y, and in general is expected to be a second order differential equation in y (see below for more comments). One is hence led to relate "sliding scale" μ to a function of y. We do this noting (1.4), and require motion in y to be associated with sliding in μ, explicitly,

$$y = \ln\mu \,, \qquad F(y) = \exp(y/L) \,, \tag{1.6}$$

where L is an arbitrary length unit.

We still need to argue how the second order differential equation in y is reduced to first order β-function equation.

NOTE: *In the above argument we focused on the marginal deformations and couplings. For relevant (or irrelevant) deformations, one would expect to see a different behavior. Considering relevant operators may take us away from the fixed point around which we have* *defined our deformed/perturbed QFT. If we the fixed point is not perturbatively close, we need to revise the above picture. The irrelevant deformations will generically deform the theory in the UV and may then change the "background" space which is defined by $F(y)$. In such cases we are not necessarily dealing with a function $F(y)$ as simple as an exponential.*

To this end, let consider a simple wave equation:

$$(\partial_t^2 - \partial_x^2)\Phi = 0 \qquad \Rightarrow \qquad (\partial_t + \partial_x)(\partial_t - \partial_x)\Phi = 0\,.$$

As we see the second order differential equation is decomposed into two first order equations, one for right-moving propagation and one for left moving. Similarly, for any second order differential equation

$$(\partial_y^2 + f(y)\partial_y + g(y))\Phi(y) = 0 \qquad \Rightarrow \qquad (\partial_y + A(y))(\partial_y + B(y))\Phi(y) = 0\,, \tag{1.7}$$

for some $A(y),\ B(y)$.

▶▶ **Exercise 1.2:** *Find $A(y),\ B(y)$ in terms of $f(y)$ and $g(y)$.*

The second order equation in (1.7) has "right" and "left" propagating modes (solutions to $(\partial_y + A(y))\Phi = 0$ and $(\partial_y + B(y))\Phi = 0$).

With the above, we can simply connect first order β-function equation to the second order e.o.m (1.7): The RG flow (β-function) equation is indeed the left moving part of the e.o.m. There are two comments in order:

- We can always choose the "left moving" solution by the choice of initial/boundary conditions.
- The RG (and the sliding scale procedure of Wilson is not a full group in the mathematical sense. If we slide from scale μ_1 to μ_2 there is a sliding scale $\mu_1\mu_2$; this procedure is transitive. However, this action does not have an inverse element: we can only scale down (or move away) from the (UV) fixed point. Wilsonian RG corresponds to moving only in direction from the fixed point. In this sense it is quite natural that we are dealing with a first order diff. equation as RG flows and not a second order one. If the second order diff. equation (1.7) is viewed as e.o.m for a field the QFT only captures half of it (the "left moving part"). This choice is made by boundary conditions.
- One may wonder what the other half of solutions correspond to in the "gravity side". As we will see in these lectures they correspond to VEV of the operators $\mathcal{O}_i$.

▶▶ **Exercise 1.3:** *If the RG (1.5) is viewed as the left moving sector of e.o.m a field $\lambda(y)$ on the $d+1$ spacetime, what should the "self interactions" of the λ field be? (Note the RHS of (1.5).)*

Let us now examine the "background" metric we obtained through the above discussions:

$$ds^2 = \sigma dy^2 + e^{2y/L} dx^\mu dx^\nu\,. \tag{1.8}$$

If $\sigma = +1$, the above metric is nothing but AdS_{d+1} spacetime in Poincaré coordinates. In this coordinate system the space has a causal boundary at $y = \infty$. We will study this space and its properties in more detail in the coming sections.

▶▶ **Exercise 1.4:** *What is this space if* $\sigma = -1$*?*

NOTE: *Eq.*(1.6) *implies that "large scales" or UV in the QFT side corresponds to large* y *region, close to the AdS boundary in the "gravity side" and conversely, IR region in the QFT corresponds to small* y *region which is "center of AdS" region.*

NOTE: *As is implicit in the above discussion, the* CFT_d *described by Lagrangian* $\mathcal{L}_0$*, is expected to correspond to the background* AdS_{d+1}*. Turning on deformations in the CFT would correspond to turning on associated perturbations on the AdS background.*

NOTE: *Hereafter, we will* choose $\sigma = +1$*, simply because we want to be dealing with a "gravity theory" with only one time-like direction.*

NOTE: *Although the AdS space has a special appearance associated with logarithmic RG flows (as discussed earlier), it is not unique and gauge/gravity correspondence may very well be formulated for other "gravity backgrounds". Of course, one should always remember that gauge/gravity correspondence (on generic backgrounds) is not necessarily the "classical gravity" limit of a full-fledge AdS/CFT type "duality".*

NOTE: *There is still an interesting case with* $\sigma = -1$*: one can take all* x^μ *coordinates to be spacelike, or equivalently take the QFT (or CFT) to be Euclidean. This case, often called* dS_{d+1}/CFT_d*, is still not fully understood as a "duality", and perhaps will not be consistent duality, relating two unitary theories. It may happen that dS/CFT "correspondence" makes some sense and could be used for performing some computations on* dS_{d+1} *space using Euclidean* d *dimensional QFT's.*

IV. It is clear from the above discussion that the gauge/gravity correspondence is a wired object. It relates a QFT to a classical field theory which involves gravity in one higher dimension. Moreover, this classical field theory is expected to have infinitely many fields. This is a property not shared by any "standard" field theory. However, there is a theory which is believed to be unitary and well defined with this property: String (Field) Theory.

That is, a natural and direct extension of Wilsonian QFT already points to "string theory". Note that this is not the usual approach to string theory, as a theory of quantum gravity.

V. One may wonder if the above general "inspiring and intuitive" discussions and ideas already show a way to "complete" the gauge/gravity correspondence to a duality? A key related question is that if we stick to Wilsonian picture one may wonder whether we will ever need to "quantize" the gravity side?

In order to answer this, let us review again the basic input went into our gauge/gravity correspondence: 1) Unitarity of the QFT, 2) Wilsonian sliding scale, 3) Locality of the QFT, 4) Cluster decomposition. These assumption, and in particular the last two, can break when we deal with "heavy" (large scaling dimension) operators. Being heavy, these operators generically negligibly contribute to the dynamics, but their presence is needed for unitarity.

When do we expect to have a possible breakdown of locality and cluster decomposition in a given QFT? This latter of course depends on the details of the QFT in question and the spectrum and relative degeneracy (number) of its heavy operators. This point will hopefully become clearer in coming lectures.

Next, we note that the best studied consistent theory we know today with infinitely many fields is string/M theory.[2] String Field theory is a theory which should be quantized, i.e. λ_i should be treated as "quantum fields" of this theory. However, one should note that the expression "quantum" in the two sides of the AdS/CFT duality corresponds to totally different notions: in the QFT sides we are supposed to path-integrate over operators $\mathcal{O}_i$, while in the other on λ_i.

One may ask the question conversely: What are consequences of the realization that λ_i should "eventually" be treated as quantum fields in a "dual" (gravity) theory sense, for the Wilsonian RG picture? Does it mean that we should be able to "integrate in" (of course, as well as the integrating out procedure we have in the Wilsonian picture)?! If yes, which is suggested by the naive/intuitive picture of gauge/gravity correspondence discussed above (recall first order vs. second order equations argument), this means that the "decoupling of scales principle" which is the underlying basis for Wilsonian RG should break down at some point. This opens a very interesting and challenging question: *How should we go about modifying/imporving Wilsonian RG to allow for integrating in?* This is of course, a question posed to any QFT and may be approached from QFT viewpoint or from gravity/string theory side. This is the area the next QFT or gravity theory breakthrough may arise and in my understanding locality and cluster decomposition will have crucial roles.

2 1/N expansion

- Given a generic field theory all observables, n-p't func'ns, are functions of parameters of the theory, like couplings and spacetime dimension or rank of the gauge or global symmetry groups. In some particular limits of the parameters, however, it may happen that these observables are only functions of certain combinations of these parameters.
- As the first example, consider the vector $O(N)$ model. Let Φ be real scalars as N-vectors of the global $O(N)$ symmetry,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi)^T \partial^\mu \Phi - \frac{1}{2} m^2 \Phi^T \Phi - \frac{\lambda}{4!}(\Phi^T \Phi)^2 \,. \tag{2.1}$$

- Let us e.g. consider the one loop β-function for $N \gg 1$ (see Peskin-Schroder section 11):

$$\beta_\lambda \propto \lambda^2 N \qquad \Rightarrow \qquad \frac{dg_{eff}}{d\ln\mu} \propto g_{eff}^2 \,, \qquad g_{eff} \equiv \lambda N. \tag{2.2}$$

  That is, there is a combination of λ and N, the effective coupling, which captures the highest power of N.

[2]We note that there is another recently emerging theory with infinitely many field: Higher Spin Theories. It is not yet clear if the HST in general can be embedded in string/M-theory or not, but there are pieces of evidence indicating they might be. This is nowadays an active field of research. The $3d$ HST is conjectured to be dual to a particular large N limit of a minimal model $2d$ CFT where the number of primaries is finite (but large).

- One may show that similar structure, i.e. the effective coupling of the theory in the leading order in the large N limit is g_{eff} and not λ, appears
  - in higher loops, e.g. two loop β-function,
  - all the other amplitudes and correlation functions.

  ►► **Exercise 2.1:** *Show that the above two statements are correct. To this end it is enough to only focus on λ and N dependence of the N point functions.*

  ►► **Exercise 2.2:** *For the $m^2 < 0$ case where we have a symmetry breaking, show that the one-loop correction to the VEV of Φ is also a function of g_{eff}. Show that the loop corrections to the one-loop effective action is also a function of g_{eff}.*

- The above, among other things, also implies that to have a well defined perturbative expansion in the large N limit it is not enough to have a small λ, g_{eff} should be small.

- One may then use the above observation to arrange a new expansion for the above theory, for a generic n-point function $G(x_1, x_2, \cdots, x_n)$

$$G(x_1, x_2, \cdots, x_n) = \sum_{k=0}^{\infty} \sum_{g \geq 0} G_{k,g}(x_1, x_2, \cdots, x_n) g_{eff}^k N^{-g} \tag{2.3}$$

  such that in the large N limit only the $g = 0$ contribution remains. In other words, any correlator has a double expansion in powers of the effective coupling and $1/N$.

  ►► **Exercise 2.3:** *Compute the first $1/N$ correction to the one-loop β-function.*

- The above $1/N$ expansion can also be worked out for $SU(N)$ gauge theories. Let us start with pure YM theory

$$\mathcal{L} = -\frac{1}{4g_{YM}^2} \text{Tr}\ F_{\mu\nu} F^{\mu\nu}\,, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]\,, \tag{2.4}$$

  where the gauge field A_μ is in adjoint representation of the $SU(N)$ gauge groups; they are $N \times N$ hermitian, traceless matrices.

- Recalling the $O(N)$ vector model analysis, one would expect that here some combination of g_{YM} and N should play the role of the effective coupling. To this end, let us e.g. recall the β-function of the theory:

$$\frac{dg_{YM}}{d\ln\mu} = -\frac{11}{3(4\pi)^2} g_{YM}^3 N \qquad \Rightarrow \qquad \frac{d\lambda}{d\ln\mu} = -\frac{22}{3(4\pi)^2}\lambda^2\,, \quad \lambda \equiv g_{YM}^2 N\,. \tag{2.5}$$

- λ is called the 't Hooft coupling.

- One may then show that any n point function of gauge invariant operators admits an expansion of the form

$$G(x_1, x_2, \cdots, x_n) = \sum_{l=0}^{\infty} \sum_{g \geq 0} G_{l,g}(x_1, x_2, \cdots, x_n) \lambda^l (1/N)^g\,. \tag{2.6}$$

- 't Hooft's genius appeared in a very interesting interpretation of the above expression (see `Nucl.Phys. B72 (1974) 461`):

  For a given Feynman diagram one can readily read the power of g_{YM} from the number of gauge interaction vertices. To read the power of N, one may focus on the gauge indices and recall that any $N \times N$ matrix can be view as $M^i{}_j$ object, where i and j run form 1 to N (they respectively indicate anti-fund. and fund. $SU(N)$ indices); this object may be represented in a "double line" notation: Each field in the adjoint may be viewed as a narrow strip. A generic gauge invariant operator may be constructed by taking trace over a product of gauge fields (or more precisely the covariant derivatives D_μ), like

$$\mathcal{O} =: \text{Tr } (D_{\mu_1} D_{\mu_2} \cdots D_{\mu_n}) : \ . \tag{2.7}$$

  In general, however, gauge invariant operators can be of the form of "multi-trace operators":

$$\mathcal{O} =: \text{Tr } (D_{\mu_1} D_{\mu_2} \cdots D_{\mu_n}) \text{Tr } (D_{\nu_1} D_{\nu_2} \cdots D_{\mu_m}) \cdots \text{Tr } (D_{\alpha_1} D_{\alpha_2} \cdots D_{\alpha_k}) : \tag{2.8}$$

- In this double line notation the single trace operator is an annulus with n points on it and the multi-trace operator consists of multi-annuli.

- When we perform matrix products we are always "contracting" a fundamental index with an anti-fund. index and the propagator (two point function of gauge fields) is of the form

$$\langle A_\mu{}^i{}_j(x) A_\nu{}^k{}_l(y) \rangle = \frac{M_{\mu\nu}}{|x-y|^2} \delta^i{}_l \delta^j{}_k \, . \tag{2.9}$$

- Let us first consider the two point function of two single trace operators of the form (2.7). Using (2.9) one can readily see that in the free theory the leading power of N is n and this two point function involves powers of N, from n all the way to 0.

- Using the double line notation, the above can be view as follows: In computing the $\langle \text{Tr } (D^n)(x) \text{Tr } (D^n)(y) \rangle$ the first two contractions just "opens up" the two annuli and we remain with $n-1$ points on two parallel strings which should be connection to each other. This may be done "keeping the order" or the contraction can happen with a "crossing".

- It is easily seen that with each crossing we lose one power of N. This is the diagramatic origin of the $1/N$ expansion.

- If we connect the points on the string keeping the order one can draw the lines on a 2d plane (topologically an S^2), the "planar diagrams". If we have g number of crossings the lines can be drawn on a surface of genus g without crossing each other, the "non-planar diagrams". Therefore, one can view each term of fixed power of λ in (2.6) as a "genus" expansion.

- The double line notation also opens up the possibility of interpreting the lines and vertices of the gauge theory as "stringy" lines, stringy Feynman diagrams. In this viewpoint, it is then "natural" to interpret the $1/N$ expansion as expansion in power of string coupling with

$$\textit{string coupling} = 1/N \tag{2.10}$$

- String theory amplitudes, besides an expansion in powers of string coupling also admit an expansion in powers of the string scale α', such that in the $\alpha' \to 0$ limit the "stringy" effects tend to zero and string theory reduces to a (super)gravity theory. In the gauge theory n-p't func'n we have an expansion in powers of λ. Whether the α' expansion of string theory and λ expansion in gauge theory are related to each other is not clear at this stage. We will return to it later.

- As we will see this is type IIB fundamental strings on $\mathrm{AdS}_5\times \mathrm{S}^5$ which are "dual" to single trace operators in the $\mathcal{N}=4$ $4d$ SYM. The fact that string theory in $10d$ on a $\mathrm{AdS}_5\times \mathrm{S}^5$ and not e.g. flat space is dual to d gauge theory was not at all visible from 't Hooft's $1/N$ expansion. This needed new information from string theory, and in particular D-branes. We will discuss how D-branes "bridged" between string theories and gauge theories in the next sections.

  ▶▶ **Exercise 2.4:** ***String bit model.*** *In quest for finding which string theory is the one described by 't Hooft's 1/N expansion, and in particular, what is the meaning of the 't Hooft coupling, one may try to model a string as a system of balls and springs, or "string bit model". Each of the string bits may be viewed as a gauge field of the YM theory and a single string state is a single trace operator consisting of some number of "bits". The bits interact via gauge field coupling and hence string tension is related to* $g_{YM}^2 N=\lambda$. *Think more about this model and explore what else one can learn from this picture.*

- Note that $1/N$ expansion, as the above discussions indicate, works for gauge theory in any dimension.

▶▶ **Exercise 2.5:** *Show that* $1/N$ *expansion also works for the SO(N) or Sp(N) gauge theories.*
Show that $1/N$ *expansion also works beyond pure YM, e.g. for QCD-like theories with* N_f *quarks in the fundamental rep. of* $SU(N_c)$*, consider the* $N_f \ll N_c$ *case. How should we modify the* $1/N$ *expansion if* $N_f, N_c \gg 1$ *while* $N_f/N_c =$*finite.*

▶▶ **Exercise 2.6:** *Show that for correlator of multitrace operators the leading* $g=0$ *term in* $1/N$ *expansion is absent. Explicitly, consider two operators of the same scaling dimension:*

$$\mathcal{O}_1(x) = Tr\,(D_{\mu_1}D_{\mu_2}\cdots D_{\mu_n})\,,\quad \mathcal{O}_2(x) =: Tr\,(D_{\mu_1}D_{\mu_2}\cdots D_{\mu_k})\,Tr\,(D_{\mu_{k+1}}D_{\mu_{k+2}}\cdots D_{\mu_n}), \tag{2.11}$$

Then $\frac{\langle\mathcal{O}_1\mathcal{O}_2\rangle}{\langle\mathcal{O}_1\mathcal{O}_1\rangle} \sim 1/N$ *in the leading* $1/N$ *contribution. In general the more number of traces, the more powers of 1/N suppression.*

Summary:
1) In the large N limit, the effective coupling of gauge theory is $\lambda = g_{YM}^2 N$.

2) Gauge theory amplitudes admit a $1/N$ expansion. The leading large contribution is given by the planar diagrams which is parametrically proportional to only a power of λ.
3) For correlator of multitrace operators the leading large N contribution may start with non-zero (positive) powers of $1/N$.
4) In view of the similarity of $1/N$ expansion and the genus expansion of string theory, one is led to think that gauge and string theory should be somehow related; exactly how?, is not seen from the discussions above.

NOTE: *The above 1/N expansion and 1/N suppression of correlators of non-planar diagrams may be "overshadowed" by other dressing factors which may arise e.g. when we consider correlator of operators with large scaling dimensions; the correlators may be enhanced by powers of scaling dimension* Δ*, because increasing the number of fields in the operator will increase number of ways one can make (Wick) contractions. Explicitly calculations shows this "dressed expansion parameter" is* Δ^2/N*. In a similar way, the 't Hooft coupling may also be dressed, to a combination like* $g^2_{YM}N/\Delta^2$*.*
`See Rev.Mod.Phys. 76 (2004) 853, hep-th/0310119.`

NOTE: *The above* $1/N$ *expansion is indeed a property of "Trace" operators. If one builds gauge invariant operators in a different way, e.g. by taking determinant or subdeterminants, or considering the Wilson loops and so on, then double line notation in which an operator looks like an "annulus" is no longer true and hence there is no "genus expansion parameter" like* $1/N$*. It happens that for these other cases one can arrange a different 1/N-like expansion.*

3 Some basic facts about CFT's in diverse dimensions

Here we will be very brief and a detailed discussion may be found in lecture notes of my CFT course `http://physics.ipm.ac.ir/phd-courses/semester7/CFT-course-2013.pdf`.

3.1 Conformal and superconformal algebras

- The conformal symmetry is a special class of diffeomorphisms which transforms components of metric tensor up to a scale factor, i.e. $g_{\mu\nu} \to f(x)g_{\mu\nu}$.
- In other words, the conformal group is the overlap of diffeomorphisms and the Weyl scaling.
- The conformal algebra is then the algebra generated from the Lie brackets of the conformal transformations mentioned above.
- In general *Minkowski spacetime* $d > 2$ dimensions, the conformal group is hence $so(d,2)$. This is a Lie algebra associated with the conformal group $SO(d,2)$. The conformal group is the isometry group of a flat $\mathbb{R}^{d,2}$ dimensional space. More importantly, $SO(d,2)$ is the isometry group of AdS_{d+1}.

- For *Euclidean* $d > 2$ dimensional space, the conformal algebra is $so(d+1,1)$ and the associated group is $SO(d+1,1)$, which is the isometry group of $\mathbb{R}^{d+1,1}$ space. Interestingly, $SO(d+1,1)$ is the isometry group of dS_{d+1} space.

- In $d = 2$ case, the story is more interesting, the conformal algebra is infinite dimensional; In the Euclidean case and in terms of complex coordinates $z, \bar{z}$ one can show that any holomorphic diffeomorphism $z \to f(z)$ is a conformal transformation. This infinite dimensional algebra allows for a central extension $\mathbf{c}$ (unlike the $d > 2$ cases). This algebra is called "Virasoro algebra":

$$[L_m, L_n] = (n-m)L_{m+n} + \frac{\mathbf{c}}{12}n(n^2-1)\delta_{m+n}\,, \qquad m, n \in \mathbb{Z}. \tag{3.1}$$

  To be more precise, the $2d$ conformal algebra in general contains two copies of the above Virasoro algebra, one for holomorphic sector (left-movers) and one for anti-holomorphic sector (right-movers). The left and right sectors in principle may have two different central charges.

- One can show that not all the conformal transformations associated with the Virasoro generators lead to global invertible maps on $\mathbb{C}$-plane. The global part of the Virasoro algebra is generated by $L_0, L_{\pm 1}$, which form $sl(2,\mathbb{R})$ algebra. This part does not involve the central charge $\mathbf{c}$ and is associated with the $SL(2,\mathbb{R})$ group. Two copies of $SL(2,\mathbb{R})$ then form $SO(2,2)$.

- One should note that the central charge $\mathbf{c}$ do not have geometric meaning.

- Unitary IReducible Representations (UIRREP's) of conformal algebras in $d = 2$ and $d > 2$ (conformal multiplets) has been studied in some detail. The conformal multiplets, in general are infinite dimensional; they involve infinitely many fields. They start from a "primary state" which has the lowest scaling dimension in the multiplet Δ_0 and all the other states, the "descendents", are constructed upon the primary and have scaling dimensions $\Delta_0 + n$, $n = 1, 2, \cdots$.

- Unitarity of the representation imposes bounds on Δ_0 and relates is to the spin of the conformal primary and the spacetime dimension $d > 2$:

$$\textbf{Scalar}: \qquad \Delta_0 \geq \frac{d-2}{2}\,, \tag{3.2}$$

$$\textbf{Spin 1/2}: \qquad \Delta_0 \geq \frac{d-1}{2}\,, \tag{3.3}$$

$$\textbf{Vector}: \qquad \Delta_0 \geq d-1\,, \tag{3.4}$$

$$\textbf{Antisymmetric } F_{\mu\nu}\textbf{:} \qquad \Delta_0 \geq d/2\,. \tag{3.5}$$

  The last item in the above corresponds to the field strength of a vector gauge field.

- In $d = 2$ unitarity implies positivity of central charge $\mathbf{c}$ and conformal weights, $h, \bar{h} \geq 0$, where $\Delta_0 = h + \bar{h}$ and Spin $S = h - \bar{h}$.

- *As we will see later the above unitarity bounds* (3.2) *reduces to the BF bound for mass of states on an AdS_{d+1} background. This is necessitated by the AdS/CFT.*

- In dimensions higher than two, we usually need supersymmetry to guarantee existence of a conformal (attractor) RG fixed point, where we expect to find a CFT. So, one should also study UIRREP's of superconformal algebras, which are constructed from admixture of conformal rep's and the SUSY multiplets.

- Scaling dimension of the states in a superconformal multiplet is $\Delta_0 + n/2$, $n = 1, 2, \cdots$.

- The lowest lying state for a BPS (supersymmetric) multiplet is a conformal primary, as well as a SUSY chiral multiplet, and is hence called a "chiral-primary state" and the multiplet based on it is also called chiral-primary multiplet.

- The anomalous dimension of the chiral-primary states is expected to be zero, due to SUSY protection.

3.2 Conformal and superconformal Field Theories in various dimensions

- Conformal field theories are relativistic theories which enjoy conformal invariance, i.e. they have conformal algebra as their global symmetry.

  ►► **Exercise 3.1:** *What should we get if we gauge the conformal symmetry? Do we get a meaningful theory?*

- QFTs at their RG fixed point exhibit scaling symmetry (by definition).

- It has been shown/proven in 2d and recently in 4d (`http://arxiv.org/abs/arXiv:1309.4095`) that scale invariance implies conformal invariance and hence the theory found at the fixed point is a conformal field theory.

- The above statement is believed to be true in generic dimension. In any case, scale invariance is a necessary (if not sufficient) condition for conformal invariance.

- Fixed points are zeros of β-functions which are usually computed perturbatively and in loop orders and hence finding the exact value of parameters (couplings) at the fixed point is usually a non-trivial task. Nonetheless, one can make sure of existence of a fixed point having first two loops results, *if* the theory has a *weakly coupled attractor fixed point* (see **Banks-Zaks'1982**, see also section 5 of my CFT lecture notes.)

- We are usually interested in having a non-trivial, interacting, but also weakly coupled CFT. That puts strong restrictions on theories in $d > 2$. Although we also know examples of strongly coupled fixed points (like T_N theories in 4d).

- Absence of (higher order) loop corrections is usually a result of SUSY. Therefore, almost all of the known CFTs in $d > 2$ are SCFTs.

- So far, besides $d = 2$ cases, we have Lagrangian description of SCFTs in 3d and 4d. About 6d CFTs we know that they exist, as they appear in the UV fixed point of 5d Yang-Mills gauge theories, but have no explicit formulation of them.
- Known 3d and 4d SCFTs are *gauge field theories* with various amounts of SUSY; the 3d CFTs are Chern-Simons gauge theories while 4d ones are generically of Yang-Mills type. We also have freedom in choice of matter multiplets. Below we give a very brief overview of such CFTs. For a list see sections 5 and 6 of my CFT lecture notes.
- Both in 3d and 4d cases we can have conformal fixed lines rather than fixed points. These fixed lines may interpolate between strongly and weakly coupled theories.

3.2.1 A short list of CFTs in various dimensions

CFTs can exist in basically any dimension, however, as mentioned to ensure having a fixed point where the QFT is expected to flow to a CFT one usually needs supersymmetry. Supersymmetry, i.e. the superconformal algebra, then puts restrictions on the number of dimensions: in higher than $6d$ the smallest superconformal multiplet will involve states of massless spin two or higher and therefore we cannot have a SCFT in $d > 6$. Classification of CFTs may be based on the dimension of the theory, as well as the amount of supersymmetry is preserves.

▶ $2d$ **CFTs**

$2d$ CFTs are the most extensively studied CFTs. There are three reasons for this:

- As discussed, the $2d$ conformal algebra, the Virasoro algebra, is infinite dimensional and $2d$ CFTs are among integrable models.
- In $2d$, unlike any other dimension, scalar and gauge fields have vanishing scaling dimension. This opens the possibility of constructing infinitely many CFTs in a very simple way, e.g. any combination like $G_{ij}(\Phi)\partial\Phi^i\bar{\partial}\Phi^j$, for any function G_{ij} and any number of fields Φ^i can potentially define a CFT (one should of course still check absence of Weyl anomaly. Although this restricts form of G_{ij}, we still remain with infinitely many choices.) In a more technical language, in $2d$ we have infinitely many marginal $(1,1)$ operators, i.e. operators with dimension one, both in left and right moving sectors, which could be added to the Lagrangian and take a given $2d$ CFT to another $2d$ CFT.
- One may also add to the above the fact that string worldsheet theory is/should be a $2d$ CFT. As such $2d$ CFTs were also developed in connection with string theory interests. This latter included an extensive study and classification of $2d$ CFTs with various amounts of SUSY. In $2d$, where we have Majorana-Weyl one component fermions, the SUSY algebra is labeled by $(\mathcal{N}_L, \mathcal{N}_R)$. Almost all combinations with $\mathcal{N}_L, \mathcal{N}_R \leq 4$ has been analyzed. For a detailed discussion on these theories, see the two volume string theory books by *Joe Polchinski*, and the CFT book by *di Francesco et al.*

▶ $3d$ **CFTs**

- Analysis of $3d$ CFTs was intensified in the last 8 years, and especially after the work of ABJM, where as explicit Lagrangian description for such theories was presented. We now know of many $3d$ CFTs which all of them are of the form of Chern-Simons gauge theories coupled to matter.

- To ensure absence of non-zero β-functions we need to add supersymmetry too. The generic structure of these theories are of the form of $G_1 \times G_2$ Chern-Simons gauge theory (with G_1 or G_2 can be $SU(n)$, $Sp(n)$ with possibly additional $U(1)$ for the case with $\mathcal{N} = 6$ SUSY, see e.g. `arXiv:0807.1102`). The matter fields are in the bi-fundamental rep's of G_1 and G_2.

- $3d$ Chern-Simon-matter CFTs can have $\mathcal{N} = 3, 4, 6, 8$. (Note the $\mathcal{N} = 8$ is the maximum supersymmetry allowed in $3d$.) The corresponding supersymmetry algebra is $Osp(\mathcal{N}|4)$. The bosonic part of the algebra consists of $sp(4) \simeq so(3, 2)$, the $3d$ conformal algebra, as well as the $so(\mathcal{N})$ R-symmetry group. (See my CFT lecture notes, section 6 for more discussion.)

- $3d$ SCFTs may also be obtained from $3d$ super-Yang-Mills (SYM) theory at their IR fixed point. The theory is strongly coupled at IR fixed point.
  ►► **Exercise 3.2:** *Argue why this is the case.*
  Nonetheless, there are string/M theoretic arguments that such a fixed point exists, and we now know that it is described by a Chern-Simons-matter gauge theory.

► $4d$ **CFTs**

All the known examples of $4d$ CFTs, to my knowledge, are of the form of (or related to) Yang-Mills (YM) gauge theories, and most of them are in fact supersymmetric, SYM theories.[3] This is related to the simple fact that in $4d$ the gauge coupling is dimensionless. To have a vanishing β-function (and desirably) weakly coupled fixed point, we need to add "appropriate" amount of matter fields. Generically, the matter fields should be such that the one-loop β-function is negative while the two-loop β-function is positive and slightly bigger than the one-loop result, so that β-function can vanish (see *Banks-Zak 1982*, and section 5 of my CFT lecture notes). This is indeed possible for appropriate N_f/N_c ratios.

Since $4d$ CFTs are mainly of the form of SYM theories, one may classify them by the amount of SUYS:

- $\mathcal{N} = 4$ **SYM**: This theory has the largest amount of supercharges with gauge multiplet as the smallest representation of the SUSY algebra. This theory cannot be deformed by any "matter fields" without reducing the amount of SUSY. The $\mathcal{N} = 4$ SYM is hence specified by only two parameters: the gauge coupling g_{YM} and the gauge group G. G can be $SU(N)$ or $\prod_i SU(N_i)$.

  The $\mathcal{N} = 4$ gauge multiplet contains a vector gauge field A_μ, four Weyl fermions ψ_a and six real scalars ϕ_I. These fields, being in the same SUSY multiplet, should all be in the same representation of the gauge group G, as the gauge field A_μ. That is, they

[3]From the known examples those which are perturbatively accessible SCFT are all of the form of SYM.

are all in the adjoint rep; for the $SU(N)$ gauge group, these are all $N \times N$ unitary traceless matrices. When $G = \prod_i SU(N_i)$ the fields can be in the bifundamental of the gauge groups. The $\mathcal{N} = 4$ $SU(N)$ SYM has a vanishing β-function for any value of g_{YM} and is hence a SCFT (for any value of g_{YM}).[4]

►► **Exercise 3.3:** *Can we have an $\mathcal{N} = 4$ $SO(N)$ or $Sp(N)$ SYM?* ***Hint:*** *Recall possible representations of these groups.*

- $\mathcal{N} = 2$ **SYM**: In this case the gauge theory may be constructed from a gauge multiplet and a hypermultiplet. The hypermultiplet can be in the fundamental, adjoint or other representations of the gauge group. The β-function of the theory is one-loop exact and hence existence of conformal fixed point means vanishing one-loop β-function. For the simple gauge groups, like e.g. $SU(N_c)$, β-function vanishes if $N_h = 2N_c$ for N_h number of hypermultiplets in the *fundamental* rep. of $SU(N_c)$, and $N_h = 1$ if the hypermultiplet is in the adjoint. (The latter has the same matter content of an $\mathcal{N} = 4$ SYM.)

  ►► **Exercise 3.4:** *Analyze and discuss the $\mathcal{N} = 2$ theory with a product of $SU(N)$'s gauge group.*
  What about $SO(N)$ or $Sp(N)$ gauge theories?

  ►► **Exercise 3.5:**

- $\mathcal{N} = 1$ **SYM**: In this case, the theory is constructed from gauge and chiral multiplets and the β-function is two-loop exact. Chiral multiplets may be in fundamental or adjoint rep's of the gauge group, which in principle can be any compact (with positive-definite metric) Lie algebra. For the case of $\mathcal{N} = 1$, QCD-like theory, i.e. $SU(N_c)$ gauge group with N_f chiral multiplets in fundamental rep of $SU(N_c)$, when $3/2 \leq N_f/N_c \leq 3$ falls in the "conformal window" the theory flows to a CFT in the IR.

- For the generic $\mathcal{N}$ CFT, the superconformal algebra is $SU(2,2|\mathcal{N})$. The bosonic part of the algebra contains $su(2,2) \simeq so(4,2)$ as well as the R-symmetry group, which is $su(4)$ for $\mathcal{N} = 4$;[5] $su(2) \times u(1)$ for $\mathcal{N} = 2$; and $u(1)$ for $\mathcal{N} = 1$.

- One may in general start with an $\mathcal{N} = 4$ or $\mathcal{N} = 2$ SCFT and deform it by appropriate marginal (or even relevant) operators and obtain or flow to another $\mathcal{N} = 2$ or $\mathcal{N} = 1$ SCFT.

► $6d$ **CFTs**

Here our knowledge is very limited and we do not have a Lagrangian description of these theories. We know of their existence because of String/M theory considerations related to NS5 or M5 branes.

[4]If we require the theory to be a CFT, then the $\prod_i SU(N_i)$ gauge group option is ruled out, because in principle the theory admits mass deformation (for fields in the bifundamental rep. of the gauge group). In other words, one may start with a $U(\sum_i N_i)$ $\mathcal{N} = 4$ SYM and give VEVs to scalars such that the $N \times N$, $N = \sum_i N_i$ matrices become block diagonal with blocks of size N_i. In this way we obtain an $\prod_i U(N_i)$ $\mathcal{N} = 4$ SYM theory, which is not a SCFT.

[5]To be more precise, for the $\mathcal{N} = 4$ case the superconformal algebra is $PSU(2,2|4)$.

- The $6d$ CFTs cannot be Yang-Mills gauge theories because the YM coupling is dimensionful (with mass dimension -1). However, they are related to $5d$ SYM in its strongly coupled UV fixed point, $6d$ CFT is the UV completion of $5d$ SYM.
- $6d$ CFTs can appear with $\mathcal{N} = (0,1)$ or $\mathcal{N} = (0,2)$ supersymmetries.[6]
- The conformal supersymmetric-primary multiplets are tensor multiplets, which for the $\mathcal{N} = (0,1)$ case, it is a anti-symmetric self-dual two-form field and one real scalar plus corresponding spinors. The field content of the $\mathcal{N} = (0,2)$ SCFT consists of a self-dual two-form, five real scalars and two $6d$ Weyl fermions.
- The superconformal group of the $6d$ SCFTs are $Osp(6,2|2\mathcal{N})$. The bosonic part of which contains $SO(6,2)$ (the $6d$ conformal group) and $sp(2\mathcal{N})$ the corresponding R-symmetry group.
- Finding a Lagrangian description for $6d$ CFTs and in particular $\mathcal{N} = (0,2)$ theory is an interesting and important theoretical question and challenge.

3.2.2 Dynamical implications of conformal invariance:

- The spacetime dependence of two point function of conformal primaries or their descendents is fixed by the scaling and Poincaré invariance. The rest of the two point function (other than the spacetime dependence) also fixes the metric on the Hilbert space of the theory.
- In general conformal invariance fixes the spacetime dependence of three point functions of primary operators.
- Three p't func'n of descendents may also be read from those of primaries.

  ►► **Exercise 3.6:** *Show explicitly how the above happens.*
- Four point function of primaries can be reduced to three point functions using *conformal blocks*. See section 4.9.2 of my CFT lecture notes.
- Spacetime dependence of higher n point functions are only restricted to be functions of $n(n-3)/2$ conformal-ratios.

4 A quick look through string and brane theory

String theory, in itself, is a one year independent course and what we intend to do here in this section is to give the minimum needed from string theory to make connection to decoupling limit and the AdS/CFT.

[6]The $\mathcal{N} = (1,1)$ case corresponds to the $6d$ SYM with 16 supercharges. This is the theory which resides on D5/NS5 branes of type IIB and is S-duality invariant.

4.1 Fundamental strings, their spectrum and critical string theories

- String theory starts with formulation of dynamics of one dimensional objects (fundamental strings) moving in a D dimensional target space.
- As strings move in spacetime they sweep a two dimensional surface, their worldsheet. The worldsheet theory for strings is hence a $2d$ QFT. In fact, we strings we should necessarily require that this theory is a $2d$ CFT.

  ▶▶ **Exercise 4.1:** *Why do we need to require the worldsheet theory to be a CFT?*

- The properties of target space, like its metric, appear as "couplings" in the worldsheet theory, $(\mu, \nu = 0, \cdots, D-1)$:

$$\begin{aligned} S = \frac{1}{4\pi\alpha'} \int_\Sigma d^2z\ G_{\mu\nu}(X)\partial X^\mu \bar{\partial} X^\nu + B_{\mu\nu}(X)\partial X^\mu \bar{\partial} X^\nu + \Phi(X)\mathcal{R} \\ + \frac{1}{2\pi\alpha'} \int_{\partial\Sigma} dx\ A_\mu(X) n \cdot \partial X^\mu \,. \end{aligned} \tag{4.1}$$

  where Σ is the worldsheet which is taken to be a 2d Euclidean surface parameterized by complex coordinates $z, \bar{z}$ and $\mathcal{R}$ is its curvature.

  The term in the second line is the worldsheet boundary term, which is a one dim. line parameterized by x and n is the 2d vector normal to the boundary. A_μ shows the boundary interaction term for open strings.

- Equations of motion for X^μ for generic couplings $G_{\mu\nu}(X),\ B_{\mu\nu}(X)$ and $\Phi(X)$ are non-linear and not solvable. What is usually done is to fix the target space "background" to be flat space, $G_{\mu\nu} = \eta_{\mu\nu},\ B_{\mu\nu} = 0, \Phi = \Phi_0 = \textit{constant}$. Then, deviations from flat target space may be viewed as interactions of strings with the background.
- String theory on flat space is hence described by *two* parameters, α' which determines the energy units of the theory, and the string coupling g_s given by the value of dilaton field, $g_s = e^{\Phi_0}$.
- E.o.M for perturbative strings on the flat background is

$$\partial\bar{\partial} X = 0 \,.$$

  ▶▶ **Exercise 4.2:** *Show that the* $\Phi\mathcal{R}$ *does not contribute to e.o.m.*

- To this e.o.m. one should add boundary conditions to find the solution. Generically we have two such choices: closed strings and open strings. If $z = e^{\tau + i\sigma}$, then we have the "standard" possibilities:

$$\begin{aligned} \text{Closed strings :} \quad & X^\mu(\sigma, \tau) = X^\mu(\sigma + 2\pi, \tau)\,, \\ \text{Open strings,} \quad \textit{Neumann} : \quad & \partial_\sigma X^\mu(\sigma = 0, \tau) = \partial_\sigma X^\mu(\sigma = \pi, \tau)\,, \\ \textit{Dirichlet} : \quad & X^\mu(\sigma = 0, \tau) = X^\mu(\sigma = \pi, \tau)\,. \end{aligned} \tag{4.2}$$

- Closed string mode expansion:

$$X^\mu = x^\mu + p^\mu \tau + \sum_{n\neq 0} \frac{1}{n} \left[\alpha_n^\mu e^{n(\tau+i\sigma)} + \tilde{\alpha}_n^\mu e^{n(\tau-i\sigma)} \right] , \tag{4.3}$$

where α_n^μ and $\tilde{\alpha}_n^\mu$ are respectively left and right movers creation-annihilation operators:

$$[\alpha_n^\mu, \alpha_m^\nu] = n\eta^{\mu\nu}\delta_{m+n} , \quad [\tilde{\alpha}_n^\mu, \tilde{\alpha}_m^\nu] = n\eta^{\mu\nu}\delta_{m+n} , \quad [\alpha_n^\mu, \tilde{\alpha}_m^\nu] = 0 . \tag{4.4}$$

- Open string mode expansion

$$\begin{aligned} Neumann: \quad & X^\mu = x^\mu + p^\mu\tau + \sum_{n\neq 0} \frac{1}{n} \alpha_n^\mu e^{n\tau} \cos\sigma , \\ Dirichlet: \quad & X^\mu = x^\mu + \sum_{n\neq 0} \frac{1}{n} \tilde{\alpha}_n^\mu e^{n\tau} \sin\sigma . \end{aligned} \tag{4.5}$$

  As we see open strings have only one mode, the left and right movers are related to each other by the boundary conditions. Note also that the Dirichlet modes do not have the "center of mass" momentum.

- The condition of absence of (worldsheet) Weyl anomaly for flat target space fixes the number of spacetime dimensions to be 26 for bosonic (closed) string theory. As such bosonic string theory in 26 dimension is called "critical string theory". One may also study non-critical strings (in dimensions other than 26), but they should be formulated on nontrivial backgrounds.

- *Spectrum of strings.* From the above mode expansions and the conformal invariance condition of the worldsheet one may read the "spectrum" of strings, i.e. finding a relation between $p_\mu p^\mu$ and the oscillator modes. One will get a different spectrum for closed and open strings:

$$\begin{aligned} \text{Closed}: \quad & p^\mu p_\mu = \alpha' M^2 = 2\sum_{i,j=1}^{24} \sum_{n>0} \delta_{ij} \alpha_{-n}^i \alpha_n^j - 2 , \quad \eta_{\mu\nu}\alpha_{-n}^\mu \alpha_n^\nu = \eta_{\mu\nu}\tilde{\alpha}_{-n}^\mu \tilde{\alpha}_n^\nu , \\ \text{Open}: \quad & p^\mu p_\mu = \alpha' M^2 = \sum_{i,j=1}^{24} \sum_{n>0} \delta_{ij} \alpha_{-n}^i \alpha_n^j - 1 , \end{aligned} \tag{4.6}$$

  The condition $\eta_{\mu\nu}\alpha_{-n}^\mu \alpha_n^\nu = \eta_{\mu\nu}\tilde{\alpha}_{-n}^\mu \tilde{\alpha}_n^\nu$ which tells us that left and right movers should contributed equally to the mass of the state, is called *level matching condition*.

- The constants -2 and -1 are in fact coming from the zero point energies of oscillators (after quantization, which has been implicitly carried out). These are the values for $D = 26$ critical strings. Therefore, the lowest mode of strings, when the oscillators are turned off, both open and closed strings have tachyon in their spectrum.

- Presence of tachyons in the bosonic string modes needs a remedy, which is usually considering the superstring theory. This latter, may be done in two different ways: 1) addition of worldsheet fermions and carrying out GSO projection (*the Ramond-Nuevu-Schwarz*, or *RNS formulation*), 2) by a direct addition of spacetime fermions, the *Green-Schwarz formulation*.

  In either case the string theory will be *consistent*[7] on a flat space background with constant dilaton Φ, only in *ten dimensional* target space. Such string theories are called *critical superstrings*.

- One may go through the details of the superstrings mass spectrum but we will cut the long (and interesting) story short and just present the results:

  - *Open strings* with *Neumann* boundary conditions have a massless state which is a spacetime *vector*, while those with *Dirichlet* boundary conditions have a massless scalar. There are also corresponding massless fermions (forming SUSY gauge multiplets in theories with 16 supercharges).

    All the other modes are massive. Their masses are of the order $1/\ell_s$, where $\ell_s^2 = \alpha'$.
  - *Closed strings* in type II theories have the massless spectrum with $\mathcal{N} = (1, 1)$ SUGRA multiplet (for type IIA) and with $\mathcal{N} = (0, 2)$ SUGRA multiplet (for type IIB). For the type I theory we have the option of having massless spectrum of $\mathcal{N} = (0, 1)$ for massless states. We also have the infinite tower of massive states which are the same for type IIA and IIB theories. Massive states of type II and type I theories are all a multiple of $1/\ell_s$.

- Spectrum of IIA massless states:

  - **NSNS sector:** $G_{\mu\nu},\ B_{\mu\nu},\ \Phi$;
  - **RR sector:** $C_\mu,\ C_{\mu\nu\alpha}$;
  - **Fermionic sector:** two dilantinos λ_a and two gravitinos ψ_a^μ (of opposite chirality).

- Spectrum of IIB massless states:

  - **NSNS sector:** $G_{\mu\nu},\ B_{\mu\nu},\ \Phi$;
  - **RR sector:** $C_0 \equiv \chi, C_{\mu\nu},\ C_{\mu\nu\alpha\beta}$ (with self-dual field strength);
  - **Fermionic sector:** two dilantinos λ_a and two gravitinos ψ_a^μ (of the same chirality).

- Spectrum of type-I massless states:

  - **Bosonic sector:** $G_{\mu\nu},\ \tilde{B}_{\mu\nu},\ \Phi$;
  - **Fermionic sector:** one dilantino λ_a and one gravitino ψ_a^μ

[7] Consistency means cancelation of Weyl anomaly of the worldsheet.

▶▶ **Exercise 4.3:** *Count the number of propagating modes in the NSNS, RR and fermionic sectors of each of the above three* $10d$ *supergravity multiplets.*

- Corresponding to each of the above three we have a $10d$ SUGRA, i.e. type IIa, IIb and type I SUGRAs.
- In $10d$ there are two other $\mathcal{N}=1$ SUGRAs which are free of chiral anomaly, these are $HetSO(32)$ and $HetE_8\times E_8$. The latter two are associated with the corresponding heterotic string theories. There is also a "massive $10d$ SUGRA". (For more details see Vol.2 of Polchinski's string theory book.)
- The action for the bosonic part of the type IIa SUGRA is

$$\begin{aligned} S &= S_{NS}+S_R+S_{CS}\,,\\ S_{NS} &= \frac{1}{(2\pi)^7\ell_s^8}\int d^{10}x\,\sqrt{-g}e^{-2\Phi}\left(R+4\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}|H_3|^2\right)\,,\\ S_R &= -\frac{1}{2(2\pi)^7\ell_s^8}\int d^{10}x\,\sqrt{-g}\left(|F_2|^2+|\tilde{F}_4|^2\right)\,,\\ S_{CS} &= -\frac{1}{2(2\pi)^7\ell_s^8}\int B_2\wedge F_4\wedge F_4\,, \end{aligned} \tag{4.7}$$

where $H_3=dB_2,\ F_2=dC_1,\ F_4=dC_3,\ \tilde{F}_4=dC_3-C_1\wedge H_3$.

▶▶ **Exercise 4.4:** *The above action is written in the* ***string frame****. One may bring it to a Einstein frame form by a rescaling of metric by an appropriate power of* e^Φ*. Find this scaling.*
How does the NS and R and CS sectors transform under this frame change.

- The action for the bosonic part of the type IIb SUGRA is

$$\begin{aligned} S &= S_{NS}+S_R+S_{CS}\,,\\ S_{NS} &= \frac{1}{(2\pi)^7\ell_s^8}\int d^{10}x\,\sqrt{-g}e^{-2\Phi}\left(R+4\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}|H_3|^2\right)\,,\\ S_R &= -\frac{1}{2(2\pi)^7\ell_s^8}\int d^{10}x\,\sqrt{-g}\left(|F_1|^2+|\tilde{F}_3|^2+\frac{1}{2}|\tilde{F}_5|^2\right)\,,\\ S_{CS} &= -\frac{1}{2(2\pi)^7\ell_s^8}\int C_4\wedge H_3\wedge F_3\,, \end{aligned} \tag{4.8}$$

where $H_3=dB_2$, $F_1=d\chi$, $F_3=dC_2$, $\tilde{F}_3=dC_2-\chi H_3$, $\tilde{F}_5=dC_4-\frac{1}{2}(C_2\wedge H_3-B_2\wedge F_3$.

NOTE: *The* C_4 *in type IIb is self-dual, meaning that*

$$^*\tilde{F}_5=\tilde{F}_5\,,\qquad d\tilde{F}_5=H_3\wedge F_3\,. \tag{4.9}$$

▶▶ **Exercise 4.5:** *The above action is written in the* ***string frame****. One may bring it to a Einstein frame form by a rescaling of metric by an appropriate power of* e^Φ*.*

Find this scaling. How does the NS and R and CS sectors transform under this frame change.

►► **Exercise 4.6:** *Compute the* 10*d Newton constant or the* 10*d Planck length* ℓ_P *in terms of the* ℓ_s *and* g_s*.*

- The above SUGRAs appear as the *low energy effective theory* of corresponding string theories; i.e. they lead to the same amplitudes (S-matrix elements) for the massless string scattering. Moreover, their e.o.m appear as the condition for the cancelation of the Weyl anomaly on the worldsheet.

  String scattering amplitudes have contributions from massive string states as well as string loop effects. These will respectively add α'-corrections and g_s-corrections to the above actions. These corrections involve higher powers of Reimann curvature and their supersymmetric completions.

  ►► **Exercise 4.7:** *Show that contribution of massive string modes to LEET are generically appearing as* ***higher derivative*** *terms in the action.*

4.2 D-branes, in SUGRA and in string theory

String theory besides fundamental strings also has "extended objects", D-branes. Branes, depending on the approximation we use have three kind of descriptions, *D-branes as objects in SUGRAs, D-branes in string theory and. D-branes as dynamical objects in target spacetime.*

Below we will briefly review each of these descriptions.

4.2.1 D-branes in SUGRA

- (Super)gravity theories usually come with various form fields.
- In general a p-dimensional object (electrically) couples to a $p+1$-form:

$$\mathcal{L} \supset J^{\mu_1\cdots\mu_{p+1}} C_{\mu_1\cdots\mu_{p+1}} \tag{4.10}$$

  ►► **Exercise 4.8:** *What is the dimension of the object which magnetically couples to a* $p+1$*-form (in a* D *dimensional spacetime).*

- Fundamental strings as one-dimensional objects couple to NSNS $B_{\mu\nu}$ field.
- D_p-branes are solutions to type IIa or IIb (or type I) SUGRAs which carry **one unit charge** of RR $p+1$-forms. Their solution for $p \leq 6$, *in string frame* is

$$\begin{aligned} ds^2 &= F(r)^{-\frac{1}{2}} \eta_{\mu\nu} dx^\mu dx^\nu + F(r)^{\frac{1}{2}} dx^m dx^m \,, \\ C_{p+1} &= \left(1 - F(r)^{-1}\right) dx_{\mu_1} \wedge dx_{\mu_2} \wedge \cdots \wedge dx_{\mu_{p+1}} \\ e^{2\Phi} &= F(r)^{\frac{3-p}{2}} \,, \end{aligned} \tag{4.11}$$

where $\mu,\nu = 0, 1, \cdots, p$, $m, n = p+1, \cdots, 9$ and

$$F(r) = 1 + \frac{r_0^{7-p}}{r^{7-p}}\,, \qquad r^2 = x^m x^m\,, \quad r_0^{7-p} = g_s \ell_s^{7-p} (4\pi)^{5-p} \Gamma(\frac{7-p}{2})\,. \tag{4.12}$$

For a reference e.g. see `http://arxiv.org/pdf/hep-th/9306052.pdf`.

►► **Exercise 4.9:** *Show that the above object has a UNIT charge of the* $p+1$ *form.*

►► **Exercise 4.10:** *Show that the mass density/tension of the above object is proportional to* $1/(l_s^{p+1} g_s)$.

►► **Exercise 4.11:** *What is the solution in Einstein frame?*

►► **Exercise 4.12:** *Discuss the large* r *behavior of string coupling* e^Φ *for* $p < 3,\ p = 3,\ p > 3$ *cases.*

►► **Exercise 4.13:** *Do Dp-brane solutions have curvature singularity?*

- The above Dp-brane solution is 1/2 BPS (it preserves 16 out of 32 supercharges of the flat space background).

- Given that the IIA theory has only **even** p and IIB only **odd** p, IIA theory has BPS D0, D2, D4 and D6 branes, while type IIB theory has 1/2 PBS, D(-1)-branes (D-instantons), D1-branes (D-strings), D3-brane D5 and D7-branes.

  ►► **Exercise 4.14:** *Show that if Dp-braneelectrically couples to RR* $(p+1)$*-form,* D_{6-p}*-brane magnetically couples to the same form.*

- The above solutions are asymptotically flat and they break the $ISO(9,1)$ isometry of the flat background to $ISO(p,1) \times SO(7-p)$.

  ►► **Exercise 4.15:** *The above shows a "static" brane solution. How can we construct the solution corresponding to a "moving brane" with velocity* u^m *in the directions transverse to it.*

- One may easily construct multi-D-brane solution, each located at $x_i^m,\ i = 1, 2, \cdots, N$ by "superposing" Dp-brane solutions. That is, just taking $F(r)$ to be

$$F = 1 + \sum_{i=1}^{N} \frac{r_0^{7-p}}{|x - x_i|^{7-p}}\,. \tag{4.13}$$

- Since the configuration is BPS the D-branes do not exert force on each other and hence x_i's are arbitrary.

- One may consider N coincident brane case which as far as gravity description is concerned a brane with mass density proportional to $N/(\ell_s^{p+1} g_s)$.

- **Black** p**-branes** are another class of extended objects in SUGRAs, given by the same RR-field as Dp-branes but with the following metric and dilaton:

$$\begin{aligned} ds^2 &= F_-(r)^{\frac{7-p}{8}} \left[-\frac{F_+(r)}{F_-(r)} dt^2 + dx_i^2 \right] + F_-(r)^{\frac{(3-p)^2}{8(7-p)}} \left[\frac{dr^2}{F_+(r) F_-(r)} + r^2 d\Omega_{8-p}^2 \right] \\ e^{2\Phi} &= F_-(r)^{\frac{3-p}{2}}\,, \end{aligned} \tag{4.14}$$

where $i = 1, 2, \cdots, p$ and

$$F_{\pm}(r) = 1 - \left(\frac{r_{\pm}}{r}\right)^{7-p} . \tag{4.15}$$

NOTE: *The above metric is given in the Einstein frame.*

NOTE: *These are solutions describing excitations of BPS branes, by addition of "mass" but with the same RR-charge.*

NOTE: *These black brane solutions cannot be "superposed".*

►► **Exercise 4.16:** *Compute the mass density of above black p-brane using ADM formalism.*

►► **Exercise 4.17:** *Show that for the "extremal" black p-brane solution with $r_+ = r_-$ one would recover the Dp-brane solution.*

►► **Exercise 4.18:** *One may associate a temperature to the black brane solution, like we do to black holes. Compute the temperature for the above p-brane geometry.*

4.2.2 D-branes in string theory

- We presented the description of D-branes in SUGRA. Since type II theories are closed string theories, the above gives a description of D-branes as a "condensate" of massless closed string states.

- Polchinski in his 1995 paper gave a description of Dp-branes within *string theory*:

Dp-brane is an object where end points of an open strings attach, with $p+1$ Neuman boundary conditions along the brane and $10-p$ Dirichlet boundary conditions transverse to it.

- One can consider a system consisting of N non-coincident *parallel* Dp-branes. Let us label the branes by $i, j, ... = 1, 2, \cdots, N$. Then, we have in general open strings ending on i^{th} and j^{th} branes. Since open strings are orientiable one can in principle distinguish ij string from ji string; therefore, there are N^2 number of such open strings. The relative position of branes is given by the zero mode of ij strings.

- Therefore, to completely specify the system we should specify excitation level of each of the N^2 open strings mode.

- In particular, for the *massless* open string states we have $p+1$-dimensional *gauge fields* as well as $9-p$ number of *scalars* and the corresponding spinorial superpartners. Each of these are $N \times N$ matrices, i.e. they are in adjoint representation of $U(N)$.

- In general, when the Dp-branes are not on top of each other this $U(N)$ is "broken" or "Higgsed down" to $U(1)^N$, as the lowest mass of open strings stretched between branes is proportional to their relative distance.

- In theories where we have unorientable open strings, like type I theory, and/or cases involving orientifold planes, O_p-planes, we can also produce $SO(N)$ or $Sp(N)$ gauge theories on Dp-branes.

- If we have a multi-brane system consisting of branes with various relative worldvolumes with different dimensionalities, one may repeat similar steps as above and provide an "open string description" for the system. In general, we will then have various open strings with different boundary conditions on each end.

  ►► **Exercise 4.19:** *Consider a two brane system consisting of Dp-brane along* $012\cdots p$ *and a Dq-brane along* $012\cdots q$ *with* $p > q$*. How many DD, DN or NN open strings do we need to describe this system?*

- It is possible for Dp-branes to form "marginal" or non-marginal "true" bound states. For example a Dp-brane and D_{p-2}-brane can form a bound state which has a mass density lower than the sum of the masses of ingredients. This happens due to an open string tachyon condensation.

- One should note that the above description for Dp-branes is an effective description in the "probe" approximation: Open strings are living in flat space and do not see the curvature caused by the brane, they only feel the brane through their boundary conditions.

4.2.3 Low energy effective field theory of Dp-branes

- As mentioned Dp-branes have a description in terms of open strings attached to them and hence their excitations may be viewed as various open string modes.

- One may compute scattering amplitudes (S-matrix elements) of open strings off each other, or off closed strings of the bulk (i.e. scattering of closed strings off branes).

- As in the SUGRA case, one may find a low energy effective field theory which reproduces these S-matrix elements for massless open string modes at lowest order (where the string interaction corresponds an exchange of open or closed string massless states. This low energy effective action is the Dirac-Born-Infeld (DBI) theory.

- The DBI action for a Dp-brane is a $p+1$-dimensional *supersymmetric gauge field theory* with 16 supercharges. Its dynamical fields are massless states of open strings which are consists of a gauge field, $9-p$ real scalars and corresponding fermions. The bosonic part of DBI action is:

$$S_{DBI} = \frac{1}{(2\pi)^p \ell_s^{p+1}} \int d^{p+1}x \ e^{-\Phi} \sqrt{-\det\left(\mathcal{G}_{ab} + \mathcal{F}_{ab}\right)} + \int \sum_{k=0}^{p+1} (C_k \wedge e^{\mathcal{F}})\,. \qquad (4.16)$$

  In the above

  - $a, b = 0, \cdots, p$ and x^a parameterize the Dp-brane worldvolume.

- $\mathcal{G}_{ab}$ is the induced metric on the Dp-brane worldvolume, and $\mathcal{F}_{ab}$ involves two parts, a the gauge field strength F_{ab} and pullback of the NSNS B-field on the brane, explicitly:

$$\mathcal{G}_{ab} = G_{\mu\nu}\partial_a X^\mu \partial_b X^\nu \,, \qquad \mathcal{F}_{ab} = 2\pi\ell_s^2 F_{ab} + B_{\mu\nu}\partial_a X^\mu \partial_b X^\nu \,, \tag{4.17}$$

  where $F_{ab} = \partial_a A_b - \partial_b A_a$ and X^μ in a specific choice for embedding coordinates is x^a for $\mu = 0, 1, \cdots, p$ and for $\mu = p+1, \cdots, 9$, $\ell_s^{-2} X^i = \phi^i$ denote $9-p$ scalar fields. Note that $G_{\mu\nu}$ is the string frame metric.

  ►► **Exercise 4.20:** *How does the DBI action look like when the background metric is in the Einstein frame?*

- The last term in the DBI action is called the Chern-Simons term and involves RR k-form fields. In particular, it involves the RR $p+1$-form which shows that Dp-brane carries one unit of the corresponding RR-charge density.

• Similarly to the SUGRA, the DBI action also arises directly from the string worldsheet action (4.1), as the worldsheet Weyl anomaly cancelation condition `Dai, Leigh, Polchinski, Mod.Phys.Lett. A4 (1989) 2073`.

• DBI action for any p may be supersymmetrized. It leads to a $p+1$ dimensional SUSY gauge theory with 16 supercharges.

• The DBI action enjoys the following (gauge) symmetries:

  - $p+1$ dimensional diffeomorphism invariance on the worldvolume.
  - λ-gauge symmetry: $A_a \to A_a + \partial_a \lambda$, with all the other fields unchanged.
  - Λ-gauge symmetry: $B_{\mu\nu} \to B_{\mu\nu} - 2\pi\ell_s^2 \partial_{[\mu}\Lambda_{\nu]}\,,\quad A_a \to A_a + \Lambda_a$. This is manifest because the action is only a function of $\mathcal{F}$.

• When we have N coincident Dp-branes the λ symmetry is enhanced to $U(N)$. Since the DBI action involves arbitrary power of $\mathcal{F}$ or derivatives of scalar fields, for this case we need to provide "certain ordering" of $U(N)$ matrices. Direct string theory computations indicate that the correct ordering is *symmetrized trace*, meaning that we symmetrize all $U(N)$ matrices before taking the trace.

• One may study the "low energy" limit of the DBI action by sending $\ell_s \to 0$.

  ►► **Exercise 4.21:** *Show that in the $\ell_s \to 0$ limit the DBI action reduces to a SYM theory with the Yang-Mills coupling g_{YM}:*

$$g_{YM}^2 = (2\pi\ell_s)^{p-3} g_s \,. \tag{4.18}$$

  Hint: *To show the above, assume the background 10d metric is flat Minkowski and set the B-field to zero and the dilaton to a constant Φ_0, $g_s = e^{\Phi_0}$.*

- DBI in itself is a low energy effective theory which shows interactions of massless open string fields with each other and with closed strings of the bulk. Therefore, the low energy effective theory of the Brane+Bulk system is

$$S = S_{SUGRA} + S_{DBI} \,. \tag{4.19}$$

  The above is the action describing a bulk geometry with brane probes.

- Similarly to the SUGRA action, DBI action also receives α' and g_s corrections. Note that DBI action already contains α' suppressed terms which all come as *higher powers* of $\mathcal{F}$.

  ►► **Exercise 4.22:** *Argue that the contribution of massive string modes generically involve* ***higher derivative*** *terms like* $(D^n\mathcal{F})^m$ *or* $(D^{n+1}\phi)^m$, *where* $n \geq 1$ *and* $m \geq 2$.

4.2.4 11d SUGRA and M-branes

- SUSY algebra considerations indicate that $11d$, that is $10+1$ dimensions, is the largest spacetime dimensions where the smallest SUSY multiplet involves maximum spin two states; the smallest $\mathcal{N} = 1$ SUSY reps in $11d$. Therefore, one may construct a SUGRA in $11d$.

- Again SUSY algebra considerations tell us that this SUGRA is unique, with a fairly simple matter content:

  11d metric, a three four $C_{\mu\nu\alpha}$ and one 11d Majorana fermion.

  ►► **Exercise 4.23:** *Count the number of propagating d.o.f, both in bosnic and fermionc sectors.*

- Note that there is no scalar in the spectrum, unlike $10d$ cases. Therefore, there are no parameters, no moduli, in the theory.

- The action for this theory involves only a length unit, $11d$ Planck length ℓ_P. Its bosonic part is

$$S_{11d} = \frac{1}{2\ell_P^9}\int d^{10}x\,\sqrt{-g}\left(R - \frac{1}{2}|F_4|^2\right) - \frac{1}{6}\int C_3\wedge F_4\wedge F_4\,, \tag{4.20}$$

  where $F_4 = dC_3$.

- A Kaluza-Klein reduction of above $11d$ action on a circle of radius R, produces $10d$ IIa action (4.7).

  ►► **Exercise 4.24:** *Work out the above reduction and establish the above statement. In particular show that*

$$\ell_s^2 \sim \ell_P^3/R\,, \qquad g_s \sim \left(\frac{R}{\ell_P}\right)^{3/2}\,. \tag{4.21}$$

- The above connection between 11d and 10d SUGRAs, may also be extended to full IIA string theory:
  - Fundamental strings of IIA become *membranes* or **M2-branes** in 11d (which are wrapping the KK circle);
  - D2-branes are directly lifted to M2-branes;
  - D0-branes become KK-gravitons;
  - D4-branes uplift to **M5-branes**.
- Mass density or tension of M2/M5-branes are respectively ℓ_P^{-3} and ℓ_P^{-6}.
- The theory which is obtained as uplift of IIA to 11d is called *M-theory*.
- The LEET of M-theory is 11d SUGRA and IIA at strong coupling becomes M-theory ($g_s \to \infty \Rightarrow R/\ell_P \to \infty$).

  ►► **Exercise 4.25:** *Show the latter noting* (4.21).
- The M2-brane and M5-branes may be viewed as solutions to 11d SUGRA: M2-brane is electrically charged under the 3-form $C_{\mu\nu\alpha}$, while the M5-brane is magnetically charged. Explicitly:

$$\begin{aligned} ds^2 &= F(r)^{-\frac{8-p}{9}} \left(\sum_{\mu,\nu=0}^{p} \eta_{\mu\nu} dx^\mu dx^\nu \right) + F(r)^{\frac{p+1}{9}} \left(dr^2 + r^2 d\Omega_{9-p}^2 \right) \\ C_{012} &= 1 - F^{-1} \quad \text{for M2-brane}, \quad C_{012345} = 1 - F^{-1} \quad \text{for M5-brane}, \end{aligned} \tag{4.22}$$

  where $p = 2$ for M2-branes and $p = 5$ for M5 and,

$$F(r) = 1 + c_p \left(\frac{\ell_P}{r} \right)^{8-p}, \qquad c_2 = 32\pi^2\,,\ c_5 = \pi\,. \tag{4.23}$$

- The above M-brane solutions are 1/2 BPS and hence one can superpose them (as in the D-brane case).
- We do not yet know the multiple M5-brane action; this is essentially related to the 6d $\mathcal{N} = (0, 2)$ CFT.

5 The decoupling limit

As discussed in the opening lectures it may happen that a unitary quantum theory has a decoupling limit. If it exists, it means that there is a limit which takes us to a specific corner of both the Hilbert space and parameter space of the theory where we find another unitary theory. We should point out that by unitarity in this context we mean perturbative unitarity or unitarity of the S-matrix.

5.1 General remarks on decoupling limit

- Let us start exploring whether it is "natural" to have decoupling limits within QFTs. We first note that moving on the RG flow paths does not generically correspond to a decoupling limit. However, in particular cases it may happen that when we move from a given RG fixed point to another, the perturbative degrees of freedom of the theory may change, as do the values of parameters/couplings. Nonetheless, the original d.o.f are still providing a complete basis (even though non-perturbatively). As another possibility, in a d dimensional QFT one may restrict oneself to a $d-n$ dimensional sector of it by allowing the states in the Hilbert space which have vanishing momenta in the n dimensional part. As we know, e.g. from electromagnetism, this does not restrict/confine lines of force to the $d-n$ dimensions and the theory is still inherently d dimensional (due to off-shell virtual particles).

- Kaluza-Klein (KK) reductions do not lead to interesting decoupled theories either, even if we take a strict vanishing radius limit. To be more specific let us consider KK reduction of a $d+1$ dimensional YM theory of coupling g_{YM}^2 on a circle of radius R. In d dimensions we find a YM theory of coupling $g_{YM}^2(2\pi R)$, a scalar field in the adjoint representation and a tower of massive modes with masses of order $1/R$, all in the adjoint rep of the gauge group. In the $R \to 0$ limit these modes become infinitely heavy and decouple from the dynamics. Although the d dimensional scalar+YM theory in itself is a unitary theory, to keep the coupling finite we need to take $g_{YM} \to 0$ and hence the $d+1$ dimensional theory becomes trivial and noninteracting.

  ►► **Exercise 5.1:** *One may consider a more "interesting" case: Take a $U(N)$ $d+1$ dimensional gauge theory and perform the above reduction while also taking $N \sim 1/R \to \infty$. Explore this case more closely and argue whether the d theory exhibits interesting dynamics. Does this case constitute a decoupled theory?*

- One may next consider KK reduction of Einstein gravity theory, e.g. on a circle of radius R. Although the Einstein theory in itself is not a unitary field theory (due to issues with renormalizability), again we face a similar situation as above: in the $R \to 0$ limit, the d dimensional Newton constant G_d remains finite only when the $d+1$ Newton constant G_{d+1} goes to zero (recall that $G_d \sim G_{d+1}/R$.)

- From the above it is clear that Low Energy Effective Theories (LEETs) e.g. IIa or IIb SUGRAs, or DBI action, do not provide a decoupled sector in string theory. Let us first consider the SUGRA cases. In this case in the $\ell_s \to 0$ limit all massive string modes decouple. Nonetheless the gravity theory also become trivial (vanishing Newton constant), unless we also send string coupling g_s to infinity.

  ►► **Exercise 5.2:** *How should we scale g_s to obtain a finite $10d$ Planck length?*

  In this limit, nonetheless, string coupling becomes large and string theory is non-perturbative.

- Next, let us consider the DBI action. Despite being supersymmetrizable (with maximum 16 supercharges), DBI action involves higher powers of $\mathcal{F}$ and is not renormalizable and hence non-unitary. At any given order in ℓ_s^2 one may add higher derivative counter-terms and regularize the theory, but the series of such higher derivative terms do not terminate and we need infinitely many of them. This will essentially takes us to a full string theory which is unitary.

Now let us study DBI in the $\ell_s \to 0$ limit. In this limit the string theory dynamics is only described by the massless states. Moreover, as we note (4.16), there are powers of ℓ_s in the action. Let us consider the brane in the flat background case. In the leading order in ℓ_s, we should hence only keep $\mathcal{F}^2$ terms of the determinant and drop higher powers of $\mathcal{F}_{ab}$. Similarly, from the first (induced metric) term one should only keep "the trace part" which is $\eta_{ab}\partial_a\Phi^i\partial_b\Phi^i$, $i = 1, \cdots, 9-p$. Once fermions are also included and for N coincident Dp-branes, the action hence reduces to a $p+1$ dimensional $U(N)$ SYM with 16 supercharges, with coupling (4.18).

- To have a decoupled theory on Dp-branes one should hence require: 1) 10d gravity to be non-interacting and trivial; 2) SYM has a finite coupling. Recalling that

$$\ell_P^4 \sim \ell_s^4 g_s\,, \qquad g_{YM}^2 \sim \ell_s^{p-3} g_s\,, \tag{5.1}$$

where ℓ_P in the above is the 10 Planck length. The decoupling conditions then read as

$$\ell_P \to 0\,, \quad \ell_s \to 0\,, \quad g_{YM} = \textit{finite and small}\,.$$

  - For $p = 3$ the above can be easily met.
  - For $p < 3$, to satisfy them $g_s \to 0$, which means a free string theory limit.
  - For $p > 3$, we are forced to take $g_s \to \infty$ and the string theory becomes untrustable. For $p = 4$ one would need to uplift to M-theory where we find M5-branes; for $p = 5$ we will need to go to a weakly coupled S-dual frame; and for $p = 6$ there is no decoupling at all.
  - For more detailed discussion on this see: `arXiv: hep-th/9802042`.

The most interesting case seems to be the $p = 3$ one.

NOTE: *The above decoupling limit argument was made for branes on flat background. Presence of non-trivial backgrounds can change the outcome.*

▶▶ **Exercise 5.3:** *Repeat the same "decoupling" argument for the M-brane case.*

5.2 Near Horizon limit as decoupling limit

- As mentioned, $\alpha' \to 0$ limit on the DBI action for $p \leq 5$ Dp-branes leads to a decoupled theory which is a $p+1$ dimensional SYM theory with 16 supercharges. This theory is a CFT for $p = 3$ while for $p = 0, 1, 2$ it flows to CFT in the IR and for $p = 4$ flows to a CFT in the UV (the $(0, 2)$ theory).

- On the other hand, we mentioned that the same Dp-brane system has a description as solutions in the gravity. One may wonder what happens if the same $\alpha' \to 0$ is also applied to this solution.

- In this limit the closed strings of the asymptotic flat geometry of the bulk become infinitely heavy and decouple; we remain only with SUGRA modes. Moreover, generically $10d$ Newton constant also goes to zero (*cf.* discussions above) and hence generically SUGRA also has a vanishing coupling. Nonetheless, the theory is not trivial because the background is curved and closed strings which are located at different distances from D-branes see different geometries and have different effective couplings.

- The Dp-brane solution (4.11) is the extremal limit of a more general black brane solution and is written in the coordinate system where the horizon is at $r = 0$ and $U \equiv r/\ell_s^2$ measures the energy stored in the scalars of the field theory as we move away from the brane, it is a measure of the energy available to open string probing the brane.

- The $\ell_s \to 0$ limit, while keeping U fixed is hence a low energy limit probing the locus very close to Dp-branes horizon; it is a **Near Horizon (NH)** limit.

- In the NH limit one may drop 1 in the harmonic function $F(r)$ (4.12).

- From hereon we focus on the $p = 3$ case, where the gravity solution is nonsingular. In this case, the NH metric of N D3-branes is:

$$\begin{aligned} ds^2 &= R^2\left[U^2 dx_\mu^2 + \frac{dU^2}{U^2}\right] + R^2 d\Omega_5^2\,, \\ e^{2\Phi} &= g_s^2\,, \quad R^4 = (4\pi)^2 \ell_s^4 N g_s \\ F_5 &= R^3 \mathrm{vol}(\mathrm{s}^5)\,, \end{aligned} \tag{5.2}$$

  where $\mathrm{vol}(\mathrm{s}^5)$ is the volume form of a unit radius five-sphere.

- The above is $\mathrm{AdS}_5\times\ \mathrm{S}^5$ which is a maximally SUSY solution to $10d$ type IIb SUGRA.

- To ensure that IIb SUGRA is a valid description and the α' and g_s corrections are small, one should make sure that the AdS radius R in string unit is finite (and large)

$$\text{Validity of SUGRA description} \quad \Rightarrow \quad R/\ell_s \gg 1 \ \text{ or } \ g_s N \sim \lambda \gg 1\,, \tag{5.3}$$

  where λ is the 't Hooft coupling of the SYM on the branes.

  ►► **Exercise 5.4:** *Carry out the NH limit procedure on the N M2, M5-brane solutions of $11d$ SUGRA and show that we respectively find $AdS_4\times\ S^7$ and $AdS_7\times\ S^4$. How does the AdS radius scale with the number of branes?*

- The above NH limit, among other things shows that the closed strings which are very close to the brane also survive, not only the massless sector of them, but also all the massive tower. But, these closed strings find themselves in an $\mathrm{AdS}_5\times\ \mathrm{S}^5$ space, rather than an asymptotic flat one.

  ►► **Exercise 5.5:** *Convince yourself that the above statement is true.*

- Note that in taking the NH limit the stack of D3-branes which were a localized source for the self-dual five-form has disappeared, instead we have remained with a dilocalized flux.

- In the NH limit the asymptotic flat region of the D3-brane geometry has been cut out.

- The branes were located at $r = 0$ but now this place is replaced by the Poincaré horizon of the AdS_5.

- The $U \to \infty$ region is the **causal or conformal boundary** of the AdS space. More discussions on this will follow.

- One may perform a KK reduction of IIb on the S^5. This leads to $\mathcal{N} = 4$ $SU(4)$ gauged SUGRA in five dimensions. This is the maximally SUSY gauged SUGRA in $5d$.

  ►► **Exercise 5.6:** *Show that the* $5d$ *Newton constant is*

$$G_N^{(5)} \sim \frac{R^3}{N^2} \,. \tag{5.4}$$

  Note the $1/N^2$ *suppression. Note also that* $G_N^{(10)} \sim R^8/N^2$.

- As a $5d$ geometry, the AdS_5 is the vacuum solution to $\mathcal{N} = 4$ $SU(4)$ gauged SUGRA. In the gauged SUGRA there are many scalar fields in various representations of $SU(4)$ and they have a non-trivial potential. This potential has an absolute minimum and the value of the potential at the minimum is negative. This minimum value is a parameter of the gauged SUGRA, and specifies the radius of the vacuum AdS_5 solution R. So, this solution is described by two parameters, $G_N^{(5)}$ and R, or equivalently N and R.

- The above clearly exhibits that the effective $\mathrm{AdS}_5\times$ S^5 gravity coupling (both in $10d$ and $5d$) are compatible with the expectations from 't Hooft's $1/N$ expansion.

- One may also study string worldsheet theory on the above $\mathrm{AdS}_5\times$ S^5 background.

  ►► **Exercise 5.7:** *Show that the coefficient in front of the worldsheet action is* $\sqrt{g_s N} \sim \sqrt{\lambda}$. *In other words,* $\alpha'_{AdS} \sim \lambda^{-1/2}$, *where* α'_{AdS} *is the effective string scale on AdS.*

- This confirms the expectation discussed in **Exercise 2.4**, that the 't Hooft coupling in the SYM is essentially related to a string scale. This makes connection with the above mentioned point that to suppress stringy corrections to SUGRA on $\mathrm{AdS}_5\times$ S^5 one should take $\lambda \gtrsim 1$.

- All the above arguments and existence of the decoupling limit suggests the following $\mathrm{AdS}_5/\mathrm{CFT}_4$ duality:

Taking the NH limit over N *D3-brane system leaves a* $4d$ $\mathcal{N} = 4$ $U(N)$ *SYM, which is expected to be dual to IIB string theory on the* $AdS_5\times$ S^5 *background. The two parameters of the theory are related as:*

$$\lambda^{-1} \longleftrightarrow \alpha'^2_{AdS}\,, \qquad 1/N^2 \longleftrightarrow \textit{closed string coupling}\,. \tag{5.5}$$

- Similar type of duality may be argued for, for the case M2 and M5-branes. But the correspondence is more subtle due to the fact that M-theory is less understood than string theory, and also that we know less about $3d$ and $6d$ CFTs than the $\mathcal{N} = 4$ SYM.

The rest of these lectures is about stating, establishing and employing the AdS/CFT. To this end we will discuss more some properties of AdS space and study field theory on AdS space. This is to familiarize with the background in the gravity side. In the meantime we will try to gradually add pages to the AdS/CFT disctionary.

6 More on AdS spaces

In this section we discuss geometry, isometry and causal structure of AdS or locally AdS spaces.

6.1 Metric on AdS space

AdS_{d+1} space

- is a maximally symmetric space of Minkowski signature with negative curvature:

$$\mathcal{R}_{ijkl} = -\frac{1}{R^2}(g_{ij}g_{kl} - g_{ik}g_{jl})\,. \tag{6.1}$$

  where g_{ij} is its metric;

- is a solution to $d+1$ dimensional Einstein equations with cosmological constant Λ:

$$\mathcal{R}_{ij} - \frac{1}{2}g_{ij}\mathcal{R} = -\Lambda g_{ij}\,; \tag{6.2}$$

  ►► **Exercise 6.1:** *Show that* $\Lambda = -\frac{d(d-1)}{2R^2}$.

- has $SO(d,2)$ isometry;
- and may be embedded into $\mathbb{R}^{d,2}$ space

$$\begin{aligned} -X_{-1}^2 - X_0^2 + \sum_{a=1}^{d} X_a^2 &= -R^2\,,\\ ds^2 = -dX_{-1}^2 - dX_0^2 &+ \sum_{a=1}^{d} dX_a^2\,. \end{aligned} \tag{6.3}$$

Depending on the solutions we choose for (6.3) one may adopt various coordinate systems which cover the whole or a part of the AdS space and may be given different names. Each coordinate system makes a part of the isometries manifest depending on the slicings used. Some famous AdS coordinate systems are

- Global coordinates:

$$
\begin{aligned}
ds^2 &= R^2\left[-\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\Omega^2_{d-1}\right] \\
&= R^2\left[-(1+r^2)d\tau^2 + \frac{dr^2}{1+r^2} + r^2 d\Omega^2_{d-1}\right] \\
&= \frac{R^2}{\cos^2\theta}\left[-d\tau^2 + d\theta^2 + \sin^2\theta d\Omega^2_{d-1}\right] ,
\end{aligned}
\tag{6.4}
$$

In the above $\tau \in (-\infty, +\infty)$, and when $d > 1$ then $r, \rho \in [0, \infty)$ and $\theta \in [0, \pi/2]$. For $d = 1$, the AdS_2 case, $r, \rho \in (-\infty, +\infty)$ and $\theta \in [-\pi/2, \pi/2]$.

►► **Exercise 6.2:** *Work out the coordinate transformation $\theta = \theta(r)$ which relates second and third lines in* (6.4)*.*

In global coordinates the $U(1)_\tau \times SO(d)$ part of the isometry is manifest. As we see from the last metric, AdS space, up to the conformal $1/\cos^2\theta$ factor, is of the form of Einstein static Universe, it is conformal to $\mathbb{R} \times S^d$.

►► **Exercise 6.3:** *Find the embedding in $d + 2$ space* (6.3) *which leads to the above global coordinates and work out the corresponding metric.*

►► **Exercise 6.4:** *How long does it take (in global time τ) for radial light-like geodesics to travel from the center of AdS $r = 0$ to $r = \infty$?*

►► **Exercise 6.5:** *How long does it take for a radial time-like geodesics to travel from the center of AdS at $\theta = 0$ to $\theta = \pi/2$?*

►► **Exercise 6.6:** *Study more general class of null geodesics in global AdS, those which involve angular momentum on the S^{d-1} too.*

- Poincaré patch:

$$
\begin{aligned}
ds^2 &= R^2\left[U^2 dx_\mu^2 + \frac{dU^2}{U^2}\right] , \qquad \mu = 0, 1, \cdots, d-1 , \\
&= \frac{R^2}{z^2}[dx_\mu^2 + dz^2] , \\
&= dy^2 + e^{2|y|/R} dx_\mu^2 .
\end{aligned}
\tag{6.5}
$$

►► **Exercise 6.7:** *Find the embedding leading to the Poincaré patch.*

Poincaré coordinates cover only a part of the global AdS spacetime, i.e. Poincaré patch is not geodesically complete and makes manifest the $ISO(d-1, 1)$ part of the isometry group.

As it is seen the small z region corresponds to large r region (or $\theta \sim \pi/2$ region) in the global coordinates. Similarly, $z \to 0$ region corresponds to $U \to \infty$ and/or $y \to +\infty$.

►► **Exercise 6.8:** *Study time-like and light-like radial geodesics.*

►► **Exercise 6.9:** *Study more general class of null geodesics in Poincaré patch, those which involve linear momentum on the $\mathbb{R}^{d-1}$ too.*

The large z region, corresponding to $y \to -\infty$ or small U region, is a horizon, the Poincaré horizon. The Poincaré coordinates do not cover behind this horizon; Poincaré patch is not geodesically complete.

►► **Exercise 6.10:** *Show that ∂_z is a Killing horizon at $z = \infty$.*

NOTE: *Despite of having a Killing horizon, AdS in Poincaré is not a black hole, because the space do not have an* ***event horizon****; this is just a coordinate artefact.*

►► **Exercise 6.11:** *The global and Poincaré coordinates are associated with slicing of AdS_{d+1} with S^{d-1} and R^{d-1}. Study slicing of AdS_{d+1} by $\mathbb{H}^{d-1}$, where $\mathbb{H}^{d-1}$ is the $d-1$-dimensional hyperboloid.*

►► **Exercise 6.12:** *In a similar manner Poincaré coordinates is slicing of AdS_{d+1} by $\mathbb{R}^{d-1,1}$. Analyze slicing of AdS_{d+1} by AdS_d and by dS_d.*

►► **Exercise 6.13:** *Study $AdS_{q+1} \times S^{p-q}$ as well as $AdS_{q+1} \times \mathbb{R}^{p-q}$ slicing of AdS_{d+1} ($q < p-1$).*

►► **Exercise 6.14:** *AdS sapce, like sphere and dS, can be viewed as a "quotient space". Argue that AdS_{d+1} space is the quotient $SO(d,2)/SO(d,1)$. Answer the same question about dS_{d+1}.*

6.2 Causal structure of AdS space

- As discussed null radial geodesics on AdS can reach the large r (small z) region in a finite time.
- On the other hand, the AdS space cannot be extended beyond $r = \infty$ region.
- Although $r = \infty$ is not a part of AdS spacetime it is in causal contact with the rest of spacetime: one can send and receive light signals to $r = \infty$ region. This region is hence **causal boundary** of spacetime.
- One may formally extend the AdS spacetime by the addition of its causal boundary to it (this is like a "one-point compactification, see below). But, then to define physics on it one needs to define behavior of particles and fields at the boundary.
- We usually demand that nothing should pass through the boundary. That is, we should require the energy flow through boundary to be zero.
- This may be equivalent to imposing Neumann or Dirichlet boundary conditions on the fields living in AdS.
- In global (or Poincaré) coordinates the AdS space is conformal to a cylinder (or to a flat space). The **conformal boundary** of spacetime is where this conformal factor vanishes. One may add this point to the spacetime, and "compactify" it.
- For AdS sapcetime conformal and causal boundaries coincide, but for a general space-time these two need not be the same.

►► **Exercise 6.15:** *Discuss what is the conformal and causal boundaries of de Sitter space? For which spacetimes the causal and conformal boundaries are different and for which they are the same?*

- The boundary of AdS_{d+1} space in global coordinates is $\mathbb{R}\times \mathrm{S}^{d-1}$.
- Poincaré patch covers an $\mathbb{R}^{d-1,1}$ part of the global boundary.

►► **Exercise 6.16:** *What is the boundary of AdS space in dS-slicing?*

►► **Exercise 6.17:** *The case of AdS_2 is different than the higher dimensional AdS spaces. Draw its Penrose diagram and discuss the structure of null, time-like and space-like geodesics on it.*

►► **Exercise 6.18:** ***Projective boundary.*** *Given the definition of AdS through the embedding equation* (6.3) *one may define the projective boundary as what remains of the space under $X_I \to \lambda X_I$, $\lambda \to 0$ projection. This is a space given by*

$$-X_{-1}^2 - X_0^2 + \sum_{a=1}^{d} X_a^2 = 0\,, \tag{6.6}$$

like an AdS space of vanishing radius. Solve the above equation with various slicings and discuss how the projective boundary and the conformal and causal boundaries are related to each other.

►► **Exercise 6.19:** *A spacetime which is a relative of AdS space is the Lifshitz spacetime with metric*

$$ds^2 = -r^{2z}dt^2 + r^2\sum_{i=1}^{p} dx_i^2 + \frac{dr^2}{r^2}\,, \tag{6.7}$$

where z is the Lifshitz scaling parameter and can be larger or smaller than one. The $z = 1$ case recovers the AdS in Poincaré patch. Discuss its null geodesics and work out its causal structure and Penrose diagram.

6.3 Asypmtotically AdS spaces

As we discussed and will be made more precise as we move along in these lectures, AdS/CFT prescribes that any deformation or physical process in the CFT has a correspondent/dual in the AdS side. A special class of such deformations in the AdS side which is of particular interest are stationary spacetimes which are asymptotically AdS. In this part we briefly discuss such backgrounds.

As mentioned AdS_{d+1} space is a solution to some (super)gravity theory and satisfies

$$\mathcal{R}_{\mu\nu} - g_{\mu\nu}\mathcal{R} = \frac{d(d-1)}{2R^2}g_{\mu\nu}\,. \tag{6.8}$$

One may then ask if the above equation (for a given d) has other solutions. The answer is Yes. Here we mention two asymptotic AdS static solutions to the above equation.

- **AdS-Schwarzchild.** This is a deformation of AdS_{d+1} space in global coordinates and keeps $U(1)_\tau \times SO(d-1)$ part of the AdS isometry intact. For $d \geq 3$ its metric is:

$$ds^2 = -f(r)d\tau^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{d-1}^2\,, \qquad f(r) = 1 + \frac{r^2}{R^2} - \frac{2G_N M}{r^{d-2}}\,. \tag{6.9}$$

►► **Exercise 6.20:** *Analyze the light-like and time-like radial geodesics of the above metric.*

►► **Exercise 6.21:** *Draw the Penrose diagram and causal structure of the above metric.*

►► **Exercise 6.22:** *Compute the Hawking temperature of the above black hole. Compute its ADM mass and Bekenstein-Hawking entropy. What is the relation between these. Is the free energy a monotonic function of horizon radius?*

►► **Exercise 6.23:** *One can in principle write down "Kerr-AdS" type solutions which are stationary. Explore this possibility.*

►► **Exercise 6.24:** *How about the $d = 2$, AdS_3 case? Explore the possibility of having "asymptotic AdS_3 black holes". **Hint**: These are the famous BTZ black holes.*

- **AdS-black brane solution.** This is a deformation of AdS in Poincaré patch and keeps $ISO(d-1) \times U(1)_t$ part of the AdS_{d+1} isometry intact. Its metric for $d \geq 3$ is

$$ds^2 = \frac{R^2}{z^2}[-f(z)dt^2 + \sum_{i=1}^{p} dx_i^2 + \frac{dz^2}{f(z)}]\,, \qquad f(z) = 1 - \left(\frac{z}{z_H}\right)^{p+1}\,, \tag{6.10}$$

where $f(z)$ is called the "emblackening factor" and z_H is the horizon radius.

►► **Exercise 6.25:** *Analyze the causal structure of the above solution.*

- One may give a more general form for the above two solutions which also covers a third case of hyperboloid slicing:

$$ds^2 = -f_k(r)d\tau^2 + \frac{dr^2}{f_k(r)} + r^2 d\Sigma_k^2\,, \qquad f_k(r) = k + \frac{r^2}{R^2} - \frac{2G_N M}{r^{p-1}}\,, \tag{6.11}$$

and Σ_k is a p dimensional Einstein manifold of unit radius with curvature $k = 0, \pm 1$ (i.e. $R^\Sigma_{ij} = k(p-1)g^\Sigma_{ij}$, where g^Σ, R^Σ are respectively its metric and Ricci curvature). The $k = +1$ includes a sphere (recovering the AdS-Sch'ld case of (6.9)), (6.10) is recovered in $k = 0$ cases, and $k = -1$ is a new case not discussed before.

►► **Exercise 6.26:** *Show that the $k = 0$ indeed recovers* (6.10).

7 Formal statement of AdS/CFT duality

As mentioned AdS/CFT duality, which is obtained through a decoupling limit of string/brane theory, is an equivalence between Quantum Gravity on the one hand and a CFT on the other hand. Moreover, as discussed, any local QFT may be viewed as a deformation of a theory around its RG fixed point, where we usually find a CFT. That is, any local QFT can be a deformation of a CFT by its local operators (*cf.* (1.1)). All physical information of such a QFT theory is encoded in its partition function:

$$Z_{CFT}[\lambda_i(x,\mu_0)] = \int D\Phi\ \mathrm{e}^{i\int d^dx\ \left(\mathcal{L}_{CFT}+\sum_i \lambda_i(x)\mathcal{O}_i\right)}\,. \tag{7.1}$$

where $\lambda_i(x,\mu_0)$ in (7.1) are defined at a given RG fixed point, occurring at a specific scale μ_0. It would be desirable to represent the AdS/CFT duality as an equivalence between two partition functions. Nonetheless, the quantum gravity/string theory side does not have a simple/precise partition function description, but one can still formally use or introduce it.

We need to first make sense what λ_i are in the AdS$_{d+1}$ side. As discussed, $\lambda_i(x;\mu)$ are in fact fields on the AdS side (in some particular gauge, see below for more comments). Moreover, we also discussed that the scale μ in the QFT side corresponds to a direction (holographic dimension) in the AdS-side. Let us e.g. consider the AdS in Poincaré patch (6.5) with y as its holographic dimension and let the scale μ_0 correspond to $y = y_0$. If the QFT is defined at a UV fixed point ($\mu_0 \to \infty$) this corresponds to $y_0 \sim \ln\mu_0 \to \infty$; i.e. boundary of the AdS space. One may then formally define the quantum gravity partition function with field values at the AdS boundary (or any other $y = y_0$) through the path integral

$$Z_{AdS-QGr}[\lambda_i(x,y_0)] = \int D\lambda_i\Big|_{\lambda_i(x,y_0)} \mathrm{e}^{S_{QGr}}\,, \tag{7.2}$$

where the path integral is only on the field configurations taking values $\lambda_i(x,y_0)$ at $y = y_0$. We stress again that the above is a formal description and quantum gravity may not necessarily have a path integral formulation as in (7.2).

The formal but precise **expression of AdS/CFT duality** is then

$$\boxed{Z_{AdS-QGr}[\lambda_i(x,y_0)] = Z_{CFT}[\lambda_i(x,\mu_0)]\,.} \tag{7.3}$$

NOTE: *Statement of duality or the correspondence as expressed above is still missing some important point:*

$\mathcal{O}_i$ are in general rank q tensors in d dimensional Lorentz group, the corresponding couplings $\lambda_i(x;\mu)$ are also necessarily a rank q tensor of the same group. The AdS (gravity) side, however, is defined on a one higher dimension and hence its fields are $d+1$ dimensional tensors. As discussed, these two can only be equivalent once a particular gauge (and/or diffeomorphism basis) in the gravity side is used.

For example, if $\mathcal{O}$ is a Lorentz vector J^μ the corresponding coupling A_μ will also be a vector. In the gravity side, this coupling A^μ should appear as a $d+1$ dimensional vector, let us denote it by $A^I,\ I = 0, 1, \cdots, d-1; y$. Nonetheless, if this vector is a *gauge field*, one may use $d + 1$ dimensional gauge symmetry, fix the $A^y = 0$ gauge. This will remove the extra component and retains matching.

►► **Exercise 7.1:** *Repeat the above argument when $\mathcal{O}$ is a symmetric two tensor. What is the symmetry in the bulk needed to retain the matching? How about higher rank tensors?*

▶▶ **Exercise 7.2:** *What "gauge symmetry" should the gravity side have if we are interested in deformations of the QFT by q-form operators?*

▶▶ **Exercise 7.3:** *What "gauge symmetry" should the gravity side have if we are interested in deformations of the QFT by a generic spin s operator with $s > 2$?*

7.1 Statement of gauge/gravity correspondence

As discussed the above statement of AdS/CFT duality is very precise, but perhaps not so useful, because (7.2) is a very formal expression not defined in detail. One may wonder if there are "useful" limits of it which makes the quantum-gravity side more tractable. Here we discuss the gauge/gravity correspondence, which not only leads to a "computable framework" but also opens up the possibility for extension and generalization of the duality, of course not necessarily as a full-fledged duality but a 'correspondence" (*cf.* discussion of section 1).

Any theory of quantum gravity, by definition, must contain a limit where the theory becomes classical and is described by a classical (Einstein) gravity theory, possibly coupled to many other fields. If the quantum gravity theory is a string theory of M-theory, one can define this "classical gravity" limit more precisely. As discussed in previous sections, this limit involves
1) making sure that all "stringy" modes are heavy (on AdS), namely $E^2\alpha'_{AdS} \ll 1$ where E is the energy/mass of the states involved in the physical process, i.e. these states are typically very massive and hence do not contribute to the dynamics;
2) making sure that quantum effect on the gravity (massless) modes is small, i.e. fields are slowly varying and string coupling $g_s^{eff} \sim 1/N$ is small.

In the above limit, when both α' and g_s corrections are small one may safely replace S_{QGr} in (7.2) by S_{SUGRA} (on AdS). Moreover, since quantum effects are also small the RHS of (7.2) may be well approximated, in the WKB/saddle point approximation, by the value of the exponential of classical gravity action with field values satisfying classical gravity field equations with boundary conditions $\lambda_i = \lambda_i(x, y_0)$. Explicitly,

$$\boxed{Z_{CFT}[\lambda_i(x,\mu_0)] \simeq e^{iS^{gravity}_{on-shell}}\Big|_{\lambda_i(x,y_0)}} \,. \tag{7.4}$$

The above is statement of gauge/gravity correspondence.

NOTE: *As we will see in the next section the QFT couplings $\lambda_i(x, \mu_0)$ and the gravity field values at y_0, $\lambda_i(x, y_0)$ are not exactly equal, they differ by a constant factor. This will be made more precise and explicit in* (8.35)*.*

With the above discussions there are some comments in order:

- If the CFT side is an $\mathcal{N} = 4$ $4d$ $U(N)$ SYM, the quantum gravity side is precisely specified through the decoupling limit arguments of previous sections to be type IIB superstring theory on the $AdS_5\times S^5$ background. Similarly, other $4d$ CFTs (*cf.* discussions in section 3), would correspond to type IIB string theory on $AdS_5\times M_5$, where M_5 is a compact Euclidean manifold and is specified by the CFT.

- In an $\mathcal{N} = 4$ gauge theory, or a generic $\mathcal{N}$ $4d$ CFT (*cf.* discussions of section 3), we have gauge fields, scalar and the corresponding fermionic superpartners. As discussed in the AdS-side, considering operators which involve of powers of these scalars geometrically correspond to deformations/physical processes on the S^5 or M_5 part of the geometry. So, if we are interested only in correlations of operators only involving gauge fields in the AdS side, we may safely restrict ourselves only to the AdS piece.

- One may also think of an "opposite situation": It may happen that for particular set of operators we can essentially only focus on the operators involving scalars corresponding to dynamics on the S^5 piece. Such an interesting case has been discussed and shown that for single trace operators involving only scalars of $\mathcal{N} = 4$ theory the effective dynamics of the system is described by an $SO(6)$ spin-chain.

- For maximally supersymmetric $3d$ (or $6d$) CFTs the quantum gravity side is M-theory on $AdS_4 \times S^7$ (or $AdS_7 \times S^4$). For less SUSY CFTs one may deform the S^7 or S^4 parts.

- In the low energy case these theories respectively reduce to type IIb supergravity on $AdS_5 \times M_5$ or $11d$ SUGRA on $AdS_{p+1} \times M_{10-p}$ (with $p = 3, 6$).

- As one may see from (6.5), in the Poincaré patch one can use various coordinate systems which differ in the choice of the holographic direction. In the QFT side, the freedom in choosing the holographic coordinate corresponds to using different regularization methods.

  ►► **Exercise 7.4:** *Argue that using z coordinate is more closely related to dimensional regularization, while U coordinate is more closely related to cutoff regularization. Which regularization does y coordinate correspond to?*

  ►► **Exercise 7.5:** *What is the coordinate system more appropriate for the Pauli-Villars regularization?*

- Different choice of coordinates on the AdS covers a part of the AdS boundary (where the UV CFT resides). Therefore, depending AdS coordinate slicing the dual QFT lives on different manifolds; for AdS in global coordinates the CFT lives on $R \times S^p$ and in Poincaré patch the dual CFT lives on $R^{p,1}$ and so on.

- The Wick rotation on the CFT and the gravity sides act in the same way: QFT on Euclidean space is dual to gravity on Euclidean AdS, EAdS.

►► **Exercise 7.6:** *Other useful limits?! Is there a "useful" (computable) limit where the gravity side is quantum while the CFT side is classical. Which limit is this? How would one approximate* (7.3) *in this limit?*

7.2 Counting of degrees of freedom and holography

As an interesting check for the AdS/CFT duality (and/or gauge/gravity correspondence) let us try to count degrees of freedom on both sides. The goal here is not showing an exact matching, but is to illustrate what described in section 1 in a more quantitative way through a back of envelope computation.

d.o.f in gauge theory side. Let us suppose that we have a QFT on $R \times R^{d-1}$. The number of degrees of freedom, number of states in the Hilbert space is basically infinite, but one may "regulate" it by putting the theory in a box of volume $\mathcal{V}$ and assuming the highest energy of states be Λ. Number of degrees of freedom is hence

$$\#\text{d.o.f in QFT} = (\mathcal{V}\Lambda^{d-1}) \cdot N_{species}\,, \tag{7.5}$$

where $N_{species}$ is number of species of fields we have, e.g. in a pure $U(N)$ YM theory that is $(d-2)N^2$ (factor of $d-2$ is for the gluon polarizations) and for $\mathcal{N}=4$ 4d $U(N)$ SYM it is $2 \cdot 8 \cdot N^2$.

d.o.f in gravity side. Let us suppose that we have a gravity theory on $\text{AdS}_{d+1}$of radius R and Newton constant G_N. If we accept the holographic result that in a gravitational setting maximum number of d.o.f is obtained on a $d-1$ dimensional surface, and if we choose this codimension two surface to be a constant time and z surface, and let it be close to the boundary at $z = 1/\Lambda$ (recall that radial direction z corresponds to the scale in the QFT side, and cutting z close to the boundary is like imposing a UV cutoff.) The volume of this surface is

$$A_\Lambda = \int d^{d-1}x \left(\frac{R}{z}\right)^{d-1}\Bigg|_{z=\frac{1}{\Lambda}} = R^{d-1}(\mathcal{V}\Lambda^{d-1})\,. \tag{7.6}$$

If according to the holographic expectation we assume that

$$\#\text{d.o.f in gravity} = \frac{A_\Lambda}{4G_N} = \frac{R^{d-1}}{4G_N} \cdot (\mathcal{V}\Lambda^{d-1})\,,$$

equating the gravity and QFT degrees of freedom we learn that (dropping numeric factors)

$$\frac{R^{d-1}}{G_N} = N_{species}\,. \tag{7.7}$$

Let us now apply the above formula to some known cases. When we are dealing with string or M-theory on $\text{AdS}_{d+1}\times \text{S}^{D-d-1}$ with AdS and sphere radii equal up to a numeric factor, $R_S \sim R_{AdS} = R$, then:

$$G_N \sim G_N^{(D)}/R^{D-d-1}, \tag{7.8}$$

where $G_N^{(D)}$ is the D dimensional Newton constant and hence

$$N_{species} \sim \frac{R^{D-2}}{G_N^{(D)}}\,. \tag{7.9}$$

- **$\text{AdS}_5\times$ S^5 and its dual $\mathcal{N}=4$ $U(N)$ SYM.** $D=10$ and $G_N^{(D)} \sim R^8/N^2$ therefore, (7.9) implies the expected result $N_{species} \sim N^2$.

- **AdS$_4$×S^7 and its dual** $U(N)\times U(N)$ **SUSY Chern-Simons theory.** $D=11$ and $G_N^{(D)}\sim \ell_P^9,\ R^3\sim \ell_P^3 N$. We hence obtain $N_{species}\sim N^3$ which is what checked within the ABJM theory analysis.

- **AdS$_7$×S^4 and its dual** $(0,2)$ **theory.** $D=11$ and $G_N^{(D)}\sim \ell_P^9,\ R^6\sim \ell_P^6 N$. Therefore, $N_{species}\sim N^{3/2}$ which is a prediction for the dual theory, yet to be constructed.

8 Field theory on AdS space

As pointed out the AdS/CFT duality is usually *useful* when the gravity side or the CFT side are weakly coupled and/or when quantum effects are suppressed. In particular, we discussed the limit where the gravity part has a classical or semi-classical description, described by the WKB or saddle point approximation; this is the gauge/gravity correspondence limit. In this limit the AdS or gravity side of the duality is basically reduced to gravity or a classical field theory on the AdS space while in the CFT or QFT side, the theory is strongly couple with strong quantum effects. In this regime one may use AdS/CFT as a tool for studying strongly coupled (gauge) field theories. In this section, we hence work through the technicalities needed for employing the gauge/gravity correspondence, i.e. field theory on AdS space. This field may be a scalar, spinor, gauge theory or gravity on AdS. In principle one should always consider all these field theories coupled to (Einstein) gravity on AdS. These field theories will provide prototypes of gravity-side in the gauge/gravity correspondence.

Let us start with a scalar on AdS$_{d+1}$:

$$S=\int d^{d+1}x\ \sqrt{-g}\left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi-\frac{1}{2}m^2\phi^2-V(\Phi)\right]\,, \tag{8.1}$$

where $g_{\mu\nu}$ is the background AdS metric. Note that in principle one should have added the Einstein-Hilbert Lagrangian to the above action and considered scalar field coupled to gravity. Nonetheless, we will be interested in cases where the back-reaction of the scalar fields on the background AdS metric is small. For later use, it is instructive to recall the energy-momentum tensor of Φ field:

$$T_{\mu\nu}=\partial_\mu\Phi\partial_\nu\Phi-[\frac{1}{2}(\partial_\alpha\Phi)^2+\frac{1}{2}m^2\Phi^2+V(\Phi)]g_{\mu\nu}+\beta(g_{\mu\nu}\Box-\nabla_\mu\nabla_\nu+\mathcal{R}_{\mu\nu})\Phi^2\,. \tag{8.2}$$

►► **Exercise 8.1:** *Show that the last term, the β-term is coming from variations of "conformal mass term", $\beta\mathcal{R}\Phi^2$ term, which may be added to the action, w.r.t. the metric $g_{\mu\nu}$. (Such terms are usually present once one considers semi-classical (loop) effects in the gravity side.)*

►► **Exercise 8.2:** *Show that $T_{\mu\nu}$ is conserved, i.e. $\nabla^\mu T_{\mu\nu}=0$.*

The e.o.m of the above action is hence

$$(\Box-m^2)\Phi=V'(\Phi)\,.$$

As usual one will treat the potential $V(\Phi)$ perturbatively and focus on the free field equation on AdS:

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\Phi) - m^2\Phi = 0\,. \tag{8.3}$$

The above equation, depending on the slicing used for the AdS-space takes different forms and has different solutions. Here we discuss global AdS coordinates and Poincaré patch cases separately.

8.1 Global-AdS case

In global coordinates $U(1)_\tau \times SO(d)$ part of the isometry of AdS_{d+1} is manifest and hence the fields may be labeled by the corresponding quantum numbers, explicitly,

$$\Phi = e^{i\omega\tau} Y_{lm_i}(\Omega_p)\cdot\phi_{l,\omega}(\theta)\,, \tag{8.4}$$

where Y_{lm_i} $i = 1, \cdots, [\frac{d-1}{2}]$ are $SO(d)$ harmonic functions, and $\theta \in [0, \pi/2]$ is the AdS radial coordinate. The equation for $\phi(\theta)$ is then obtained as

$$\left[\tan^{1-d}\theta\partial_\theta(\cos^2\theta\tan^{d-1}\theta\partial_\theta) - l(l+d-2)\cot^2\theta + \omega^2\cos^2\theta - (mR)^2\right]\phi_{l,\omega}(\theta) = 0\,. \tag{8.5}$$

In the above we have used the fact that eigenvalue of quadratic Casimir of $SO(d)$ is $l(l+d-2)$. One may then extract out the small θ and $\theta \simeq \pi/2$ behavior of $\phi(\theta)$ as

$$\phi_{l,\omega} = \sin^l\theta(\cos\theta)^{\Delta_\pm}\varphi^\pm_{l,\omega}\,,$$

where

$$\Delta_\pm = \frac{d}{2} \pm \frac{1}{2}\sqrt{d^2 + 4(mR)^2}\,. \tag{8.6}$$

We note that $\Delta_\pm$ are solutions to

$$\Delta(\Delta - d) - (mR)^2 = 0 \qquad \Longrightarrow \qquad \Delta_+ + \Delta_- = d\,.$$

The equation for $\varphi(\sin\theta)$ turns out to be a hypergeometric function. In the end, we find

$$\Phi = e^{i\omega\tau}Y_{lm_i}(\Omega_p)\cdot\sin^l\theta(\cos\theta)^{\Delta_\pm}\ {}_2F_1(a, b, c; \sin\theta)\,, \tag{8.7}$$

where

$$a = \frac{1}{2}(l + \Delta_\pm - \omega)\,, \qquad b = \frac{1}{2}(l + \Delta_\pm + \omega)\,, \qquad c = l + \frac{d}{2}\,. \tag{8.8}$$

The above solution is hence specified by two quantum numbers ω, l and m as an independent parameter. ω is not quantized, while $l = 0, 1, 2, \cdots$.

We still need to examine the behavior of our solution close to the boundary. There are **two** conditions which should be examined. We need to make sure that

- **the energy-momentum is conserved in the AdS.** That is, the flux of energy momentum tensor through the causal boundary is zero. This condition as we will show below leads to quantization condition for ω.

- **the field is normalizable on the AdS.** This tell us which of $\Delta_\pm$ modes are normalizable.

Energy-momentum conservation:

Energy-momentum flow is $T_{\mu\nu}n^\mu\xi^\nu$ where ξ^μ is the time-like unit vector and n_μ is the space-like unit vector normal to the boundary. Therefore, energy flow through the boundary vanishes if

$$\int_{S^p} d\Omega_p\sqrt{g}\ n_i T^i{}_0\Big|_{\theta=\pi/2} = 0\,. \tag{8.9}$$

Eq.(8.9), once we recall (8.2) takes the form

$$\tan^{d-1}\theta\,[(1-4\beta)\partial_\theta + 4\beta\tan\theta]\,\Phi^2\Big|_{\theta\to\pi/2} = 0\,. \tag{8.10}$$

▶▶ **Exercise 8.3:** *Using properties of Hypergeometric functions around* $\sin\theta\sim 1$ *that the above is satisfied iff* a, b *arguments in* ${}_2F_1(a,b,c;\sin\theta)$ *are both integer-valued.*

The above is hence satisfied if

$$|\omega| = \Delta_\pm + l + 2n\,,\qquad n=0,1,2,\cdots\,. \tag{8.11}$$

Note that ω can be positive to negative, while satisfying the above.

▶▶ **Exercise 8.4:** *The "radial AdS waves": From the above we see that the lowest possible* $|\omega|$ *is obtained for* $l, n=0$. *To gain a better intuition, write the e.o.m for the scalar field in global coordinate using* r *instead of* θ *(*cf. (6.4)*). How does the wave function look like for this case as a function of* r,τ*?*

NOTE: *As pointed out global AdS coordinates corresponds to radial quantization in the QFT side and hence* $i\partial/\partial\tau$*, and hence* ω*, corresponds to scaling dimensions. Allowed values of* ω (8.11) *appear as towers above the values determined by* $\Delta_\pm$.

▶▶ **Exercise 8.5:** *In fact one can show that states with given* l*,* n *are obtained by the action of* $SO(d,2)$ *generators on* $l=n=0$ *solution.*

1. *Verify this statement.*
2. *Which* $SO(d,2)$ *generator increases/decreases* n *by one?*
3. *This in particular implies that states with non-zero* l*,* n *are (conformal) descendents of the primary operators, and that states with a given* $\Delta_\pm$ *form lowest (highest) weight states of a conformal multiplet, they are primary states. Argue that this statement is correct.*

From the above we also learn that $\Delta_\pm$ should be real valued, recalling (8.6) this means that

$$(mR)^2 \ge -\frac{d^2}{4}\,. \tag{8.12}$$

The above is called the Breitenlohner-Freedman (BF) bound. This means that on the AdS, to some extent "tachoynic" mass is also allowed.

▶▶ **Exercise 8.6:** *Compare the BF bound with the conformal mass term for scalar on AdS.*

▶▶ **Exercise 8.7:** *What are the values of $\Delta_\pm$ for the "massless" scalar field? What are values of $\Delta_\pm$ when the BF bound is saturated?*

To summarize above:

- Once the BF bound (8.12) is satisfied we have FOUR solutions for a given l and n quantum numbers *cf.* (8.11) and given mass m; they are labeled by $\Delta_\pm$ and the corresponding frequency ω can be positive or negative. This is like saying that for a given value of momentum and direction we can have four types of linear waves (positive or negative frequency, and left or right moving). Since the equation of motion we started with is linear (in Φ) the most general solution is a linear combination of these four solutions.

- To quantize the theory in the canonical quantization method, as we always do, we should replace the coefficient expansions with creation/annihilation operators and impose canonical quantization conditions. Alternatively one may perform path integral quantization. In this case one should make sure that these modes (solutions) are normalizable; only normalizable modes should be included in the path integral. This is what we will do next.

Normalizability condition:

Given the quadratic part of any field theory, it defines a norm (which is then used to define the symplectic two from used in quantization). For scalar field with the Klein-Gordon type action, this is

$$\int_\Sigma d^dx\sqrt{-g}\ \nabla^i(\Phi\partial_i\Phi) = 0 \quad \Longrightarrow \quad \int_{b'dry} \sqrt{-g}\ n^\mu\Phi\partial_\mu\Phi = finite\,, \tag{8.13}$$

where Σ is any constant time surface in AdS and n^μ is the unit vector pointing toward the boundary. The integrand of the integral at the boundary behaves like $(\cos\theta)^X$ where $X = (2\Delta_\pm + 1) + 1 - d \geq 0$, i.e.

$$\Delta_\pm \geq \frac{d-2}{2}\,. \tag{8.14}$$

NOTE: *The above normalizability condition is exactly the unitarity bound for the scaling dimension of a scalar field in d dimensional CFT. Think and argue how these two are related to each other.*

▶▶ **Exercise 8.8:** *Repeat the energy momentum conservation and the normalizability conditions for the $d = 1$ AdS_2 case. Note that in this case we have two disconnected boundaries and these two conditions should be satisfied at both.*

Recalling (8.6), it is readily seen that
Δ_+ mode is always normalizable;
Δ_- mode is normalizable only if $(mR)^2 \leq -\frac{(d-2)(d+2)}{4}$.

Next, we recall that modes should also satisfy the BF-bound $(mR)^2 \geq -\frac{d^2}{4}$. Therefore:

- If $-\frac{d^2}{4} \leq (mR)^2 \leq -\frac{(d-2)(d+2)}{4}$ there are **two** normalizable modes and;
- if $(mR)^2 > -\frac{(d-2)(d+2)}{4}$ there is only one normalizable mode Δ_+.

NOTE: *The above implies that when the BF-bound is saturated both of modes are normalizable.*

►► **Exercise 8.9:** *For massless scalar field $m = 0$, what are $\Delta_\pm$? which modes are normalizable?*

►► **Exercise 8.10:** *As two interesting examples consider $d = 2$ and $d = 4$ cases, corresponding to 2d and 4d CFTs. Work out the BF-bound and normalizability conditions.*

►► **Exercise 8.11:** *The above analysis was carried out for scalar fields. For massive vector fields described by the action*

$$S = \int d^{d+1}x \, \sqrt{-g} \, \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} m^2 A_\mu A^\mu \right] , \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \, , \tag{8.15}$$

one may repeat the above analysis. Show that

$$\Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{(d-2)^2}{4} + (mR)^2} \, . \tag{8.16}$$

What is the BF-bound in this case? What is the normalizability bounds for $\Delta_\pm$ modes?

►► **Exercise 8.12:** *Repeat the same analysis for spinors, n-forms and spin two (gravitons). This has been carried out in* ***hep-th/9802203, hep-th/9904017.****. The result is:*

Spinors *(including $s = 3/2$ gravitinos): $\Delta = \frac{d}{2} + |mR|$.*

Spin 2 *graviton: $\Delta_- = 0$ and $\Delta_+ = d$.*

n***-forms*** *($n \neq \frac{d}{2}$): $\Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{(2n-d)^2}{4} + (mR)^2}$.*

Self-dual n***-forms*** *($n = d/2$ forms): $\Delta_+ = \Delta_- = \frac{d}{2} + |mR|$ (like spinors).*

►► **Exercise 8.13:** *What are the normalizability conditions on $\Delta_\pm$ for the above cases? Compare this normalizability condition with the unitarity bound on the scaling dimension of a similar field in d CFT.*

NOTE: *Scaling dimensions $\Delta_\pm$ (of the gauge theory side) are specified by the mass parameter m in the gravity side. In particular, "massless fields" with $m = 0$ correspond to* *marginal operators in the gauge theory side with $\Delta_+ = d$ for scalar and spin two gravitons; massless gauge field in the bulk corresponds to a current with $\Delta = d-1$ and massless spinor corresponds to a spinor operator in the gauge theory side, whose fermion bi-linear is marginal, i.e. it has scaling dimension $d/2$.*

8.2 Poincaré patch case

As mentioned on may adopt various coordinate systems on the AdS_{d+1} which cover (a part of) global cover of the space, realize a part of the $SO(d,2)$ isometry of the space and contain (a part of) the causal or conformal boundary of the space. Therefore, (8.3) will take a different form once written in different coordinate systems on AdS. Here we consider the Poincaré patch coordinates (6.5) in z, x_μ coordinate system. In this coordinate system (8.3) takes the form

$$\left[z^2\partial_\mu^2 + z^{d+1}\,\partial_z\Big(z^{1-d}\partial_z\Big) - (mR)^2\right]\Phi(x;z) = 0 \tag{8.17}$$

Noting the fact that $ISO(d-1,1)$ part of the symmetry is manifest, one may use plane-waves on x^μ to solve the above equation. Explicitly, let

$$\Phi(x;z) = z^{\frac{d}{2}}\ \mathrm{e}^{ik\cdot x}\phi_k(z)\,, \tag{8.18}$$

inserting this into (8.17) leads to

$$\left[\partial_z^2 + \frac{1}{z}\partial_z - (k^2 + \frac{\nu^2}{z^2})\right]\phi_k(z) = 0\,, \tag{8.19}$$

where

$$\nu = \sqrt{(mR)^2 + \frac{d^2}{4}}\,. \tag{8.20}$$

Solutions to (8.19), which is a Bessel equation, for $k^2 < 0$ are Bessel-K or Bessel-I and for $k^2 > 0$ are Bessel-$J_{\pm\nu}$:

$$\Phi_{\pm,k}(x;z) = \begin{cases} z^{\frac{d}{2}}\ \mathrm{e}^{ik\cdot x}(c_+K_\nu(kz) + c_-I_\nu(kz)) & k^2 > 0 \\ z^{\frac{d}{2}}\ \mathrm{e}^{ik\cdot x}J_{\pm\nu}(qz) & k^2 = -q^2 < 0 \end{cases} \tag{8.21}$$

As we see a generic solution is labeled by a continuous index which is a d vector k and $\pm$ signs. The most general solution is then a linear combination of $\Phi_{\pm,k}$, or for a given k it is a linear combination of $\Phi_\pm$. For later use and to gain more intuition about these solutions let us look at the small and large z behavior of them:

- Large z behavior (around center of AdS):

$$\Phi_\pm \sim \begin{cases} z^{\frac{d}{2}}\mathrm{e}^{\pm|k|z} & k^2 > 0 \\ z^{\frac{d}{2}}\mathrm{e}^{\pm i|q|z} & k^2 = -q^2 < 0 \end{cases} \tag{8.22}$$

Therefore, for the $k^2 > 0$ case the plus sign is not regular while the minus sign is regular with an exponential decay. For the $k^2 < 0$ both solutions have a power-law growth with an oscillatory part.

- Small z behavior (close to the boundary of AdS):

$$\Phi_\pm \sim z^{\frac{d}{2}\pm\nu} \sim z^{\Delta_\pm}\,, \qquad \Delta_\pm \equiv \frac{d}{2} \pm \nu = \frac{d}{2} \pm \sqrt{(mR)^2 + \frac{d^2}{4}}\,, \tag{8.23}$$

for both $k^2 > 0$ and $k^2 < 0$ cases. Note, perhaps expected, (re)appearance of $\Delta_\pm$ factors, *cf.* (8.6).

One should again study zero-energy flow through the boundary and the normalizability conditions. The analysis is similar to the global-AdS case. The only difference is that the radial direction θ and normal to the boundary ∂_θ, are now replaced by z and ∂_z.

- Zero energy flow condition:

$$z^{1-d}\left[(1-4\beta)\partial_z - 4\beta/z\right]\Phi^2\Big|_{z\to 0} \to 0\,. \tag{8.24}$$

Recalling (8.23) that is

$$\left[2\Delta_\pm - 4\beta(1+2\Delta_\pm)\right] z^{2\Delta_\pm - d}\Big|_{z\to 0} \to 0\,. \tag{8.25}$$

therefore, for generic β, Δ_+ mode is acceptable while Δ_- modes is not. For $2\beta = \frac{\Delta_\pm}{2\Delta_\pm+1}$ the zero-energy flow condition becomes $\Delta_\pm > (d-1)/2$, and Δ_- will also have a chance of being acceptable if $\nu < 1$.

NOTE: *Note the difference between the implications of zero-energy flow condition at the boundary for the AdS in global and in the Poincaré coordinates: In the global case this leads to quantization of ω, while in the Poincaré case this leads to exclusion of Δ_- mode, even if it is normalizable (see below).*

- Normalizability: (8.13) in this case takes the form

$$\int_{b'dry} z^{1-d}\partial_z\Phi^2\Big|_{z\to 0} = finite \quad \Longrightarrow \quad \Delta_\pm \geq \frac{d-2}{2}\,. \tag{8.26}$$

That is, Δ_+ is always normalizable while Δ_- is normalizable only if $(mR)^2 \leq 1 - \frac{d^2}{4}$. To summarize the above, the leading terms in the near boundary expansion has the form:

$$\Phi_{z\sim 0} = c_+ z^{\Delta_+} + c_- z^{\Delta_-}\,. \tag{8.27}$$

In writing the above expansion we have been a bit sloppy and it is not an expansion in power of z. A more precise expansion may be obtained noting that (8.27) is in fact two terms derived from small z expansion of

$$\Phi_k(x;z) = \left[c_+(k) z^{d/2} I_\nu(kz) + c_-(k) z^{d/2} K_\nu(kz)\right] e^{ik\cdot x}\,, \tag{8.28}$$

that is, the low z expansion contains terms with powers of z^{Δ_-+n} for $n=0,1,2$ and if ν is an integer, we will have two types of z^{Δ_+} terms. we will return to this in the next section.

►► **Exercise 8.14:** *Complete the steps leading to the above results.*

For later use we recall that

$$\begin{aligned} K_\nu(z) &= \frac{\pi}{2\sin\nu\pi}\left(I_{-\nu}(z)-I_\nu(z)\right)\,, \qquad \nu\notin\mathbb{Z}\,,\\ I_\nu(z) &= \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty}\frac{1}{2^{2k}\cdot k!\Gamma(k+\nu+1)}z^{2k}\,, \end{aligned} \tag{8.29}$$

and for integer ν, $\nu=n\in\mathbb{Z}$ case,

$$\begin{aligned} K_n(z) =& \frac{1}{2}\sum_{k=0}^{n-1}(-1)^k\frac{1}{k!(n-k-1)!}\left(\frac{z}{2}\right)^{n-2k}\\ &+(-1)^{n+1}\sum_{k=0}^{\infty}\frac{1}{2^{n+2k}k!(n+k)!}z^{n+2k}\left[\ln\frac{z}{2}-\frac{1}{2}\psi(k+1)-\frac{1}{2}\psi(n+k+1)\right]\,, \end{aligned} \tag{8.30}$$

where $\psi(x)$ is the special function ψ defined as $\psi(x)=d\Gamma(x)/dx$. As we see for the generic ν the expansion involves $z^{\pm\nu+2k}$ $k\in\mathbb{Z}$ powers, while in the integer ν case, we have $z^{\nu-2k}$ powers for $k\leq n-1$ as well as term of the form $z^{\nu+2k}\ln z,\ k\geq 0$.

►► **Exercise 8.15:** ***Saturation of the BF bound.*** *When the BF bound is saturated and $\Delta_-=\Delta_+$ and the two solutions become identical and hence* (8.3) *should have another solution too. Find this extra solution for global coordinates and Poincaré and discuss its properties and study its normalizability and zero energy flow conditions.*

8.3 Gauge/gravity made more explicit and precise

As discussed the small z region in the gravity side corresponds to the UV region in the gauge theory side and the small z expansion (8.27) of the bulk fields. This information is already contained in our statement of the gauge/gravity correspondence (7.4). In this section we will make it more explicit.

Kinematical matching. In a CFT$_d$, which is a d dimensional QFT in its RG fixed point, primary operators $\mathcal{O}$ may be labeled by their spin $\mathbf{S}$ ($SO(d-1)$ quantum numbers) and scaling dimension Δ, and descendents by two extra "level" quantum numbers like $\mathbf{L}, n$, where $\mathbf{L}$ denotes the orbital $SO(d-1)$ harmonics. In the gravity side, solutions of field equations on AdS$_{d+1}$ (e.g. in global coordinate), are also labeled by their mass mR, $SO(d-1)$ spin $\mathbf{S}$, orbital angular momentum $\mathbf{L}$ and n. So, as we see there is very good matching between operators of QFT$_d$ and field solutions on AdS$_{d+1}$. Since the $SO(d-1)$ quantum numbers and descendent level n are mapped exactly to each other in both sides, the matching would be complete and precise if we relate Δ and mR. This is done as

$$\Delta(\Delta - d) = (mR)^2 . \tag{8.31}$$

NOTE: *The AdS_{d+1} isometry group (which is the conformal group in d dimensions is $SO(d,2)$. Like any other SO-type group its representations are labeled by "generalized" $SO(d+2)$ spherical harmonic, call it $\boldsymbol{\Upsilon}_\Delta$, where the value of its second rank Casimir operator is $\Delta(\Delta-d)$. In this viewpoint m in fact defines the $SO(d,2)$ representation. Note, however, that due to non-compactness of $SO(d,2)$ Δ in this case is not an integer. Note also that if it were $SO(d+2)$ the Casimir would have been $\ell(\ell+d)$, which upon $\ell \to -\Delta$ recovers that of $SO(d,2)$.*

We discussed the more direct correspondence between QFT operators and gravity solutions. However, for a given set of quantum numbers, there are two solutions for fields on AdS: those given by Δ_- modes or those with Δ_+ modes. To make the gauge/gravity correspondence more precise and resolve this twofold ambiguity we first recall that the deformed QFT by an operator $\mathcal{O}$ with scaling dimension Δ_+ is described by

$$S_{def} = S_0 + \int d^dx \ \lambda \cdot \mathcal{O} . \tag{8.32}$$

Since $\lambda \cdot \mathcal{O}$ is a term in the action, if $\mathcal{O}$ has scaling dimension Δ_+, λ has scaling dimension $d - \Delta_+ = \Delta_-$.

Relevant, marginal and irrelevant operators

For QFT we use a standard terminology for classifying the operators by their scaling dimension to relevant, marginal or irrelevant. In the gauge gravity correspondence and for a scalar field/operator this parallels to

- A scalar field on AdS with $m^2 > 0$, corresponds to an irrelevant spin zero (scalar) operator in the gauge theory with $\Delta_+ > d$.
- A scalar field on AdS with $m^2 = 0$, corresponds to a marginal spin zero (scalar) operator in the gauge theory with $\Delta_+ = d$.
- A scalar field on AdS with $-d^2/4 \leq m^2 < 0$, corresponds to a relevant spin zero (scalar) operator in the gauge theory with $\Delta_+ < d$.

►► **Exercise 8.16:** *Repeat the above for spinor, vector and n-from fields.*

To make use of usual QFT notions, it is more convenient to work with QFT on R^d (rather than $R \times S^{d-1}$) and hence we focus on field solutions on AdS in Poincaré patch. These solutions are labeled by d vector k and $\Delta_\pm$. Next we recall that since Φ is a scalar field and we are working in a coordinate system where all coordinates are dimensionless, then

$$\Phi_k(x;z) = \Phi_{\lambda k}(\lambda^{-1}x; \lambda^{-1}z) ,$$

for an arbitrary constant λ. Therefore, under the above scaling

$$c_\pm \to \lambda^{\Delta_\pm} c_\pm \,. \tag{8.33}$$

That is, $c_\pm$ have scaling dimension $\Delta_\pm$ and hence it is natural to associate c_- with the coupling λ (which has scaling dimension Δ_-) and c_+ with $\mathcal{O}$ which has scaling dimension Δ_+. From (8.27) we learn that the leading contribution to Φ around $z = 0$ is coming from c_- term, i.e.

$$c_- = \lim_{z\to 0}(z^{-\Delta_-}\Phi)\,. \tag{8.34}$$

With the above we can now make (7.4) more precise

$$\boxed{Z_{CFT}[\lambda_i(x;\epsilon)] \simeq e^{iS^{gravity}_{on-shell}}\Big|_{\lambda_i^{gravity}=\epsilon^{\Delta_-}\lambda_i(x,\epsilon)}\,,} \tag{8.35}$$

where in the last equality, $\lambda_i^{gravity}$ is the value of the field in the gravity side in the small ϵ limit which is equal to the value of the coupling of the field theory operator $\mathcal{O}$, $\lambda_i(x;\epsilon)$. Note that ϵ in the gravity side corresponds to the value of the holographic z coordinate close the the boundary, while in the gauge theory side it corresponds to the cutoff in dimensional regularization.

NOTE: *The above makes it explicit that the two differ by an ϵ^{Δ_-} power (*cf.* the comment below* (7.4)*). It is notable that for massless scalar and spin two cases $\Delta_- = 0$, where the dual operator $\mathcal{O}$ is marginal, the two become exactly equal.*

We are now ready to perform explicit computations from (8.35), e.g.

$$\begin{aligned}
\langle\mathcal{O}(x)\rangle &= \frac{1}{\sqrt{g_{bdry}}}\frac{\delta S_{ren}}{\delta\phi_0(x)}\Big|_{\phi_0=0}\,,\\
\langle\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle &= -\frac{1}{\sqrt{g_{bdry}}}\frac{\delta^2 S_{ren}}{\delta\phi_0^1(x_1)\delta\phi_0^2(x_2)}\Big|_{\phi_0=0}\,,\\
\langle\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle &= (-)^{n+1}\frac{1}{\sqrt{g_{bdry}}}\frac{\delta^n S_{ren}}{\delta\phi_0^1(x_1)\cdots\delta\phi_0^n(x_n)}\Big|_{\phi_0=0}\,,
\end{aligned} \tag{8.36}$$

where S_{ren} is on-shell gravity action after divergent terms in ϵ has been extracted out (more discussions on this will follow), g_{bdry} is the metric at the d dimensional causal boundary of asymptotic AdS$_{d+1}$ and $\phi_0^i(x)$ is the source for the operator $\mathcal{O}^i(x)$.

According to gauge/gravity dictionary

- c_-, the coefficient of the non-normalizable mode (at $z = 0$), corresponds to the value of coupling of the operator $\mathcal{O}$ in the UV. Absence of non-normalizable mode, $c_- = 0$ choice, would hence correspond to deformation of the theory with vanishing coupling in the UV, where the fixed point is and one would recover the CFT.

- c_+, the coefficient of the normalizable mode in (8.27), corresponds to the VEV of the operator $\mathcal{O}$ in the UV. This latter, as we will see next, may be directly derived from (7.4).

NOTE: *Generically we choose* $c_- = 0$, *the* standard quantization in AdS/CFT, *because we want the dual gauge theory theory to flow to a fixed point in the UV. For* $(mR)^2 \le 1 - \frac{d^2}{4}$ *we have other choices for the gauge/gravity dictionary: we may also choose* $c_+ = 0,\ c_- \neq 0$, *the* alternate quantization in AdS/CFT. *This corresponds to relating* c_+ *to the coupling and* c_- *to the VEV of the operator.*

8.3.1 Boundary to bulk propagator

As discussed the most general solution for free (scalar) fields on the AdS background is specified by two class of quantum numbers.

- In the global coordinates case, which corresponds to the *radial quantization* in the QFT side, these are n, l, m_i and there are two modes for a set of $(n; l, m_i)$: normalizable and non-normalizable.
- In the Poincaré patch, these quantum numbers are replaced by d-vector k, and again there are two modes $\Delta_\pm$ for a given k.

In both cases, since the equation is linear the most general solution is a linear combination of the above. Let us focus on the Poincaré patch case. The most general solution is hence of the form:

$$\Phi_{generic}(x; z) = \int d^d k\ \left[\phi_+(k; z) e^{ik\cdot x} C_+(k) + \phi_-(k; z) e^{ik\cdot x} C_-(k)\right] \tag{8.37}$$

where $\phi_\pm(k; z)$ are related to $\Phi_\pm$ (8.21) and $C_\pm(k)$ are arbitrary functions of k.

- If we demand the normalizability and zero-energy-flow conditions, only the Δ_+ mode is allowed. We may then set $C_-(k) = 0$ and $\Phi_{generic}$ is then completely specified by giving function $C_+(k)$.
- Instead of the function $C_+(k)$ one may use its Fourier transform $\varphi(x)$. Note that this function has now z dependence.
- Next, we note that (when $C_- = 0$)

$$z^{-\Delta_+} \Phi_{generic}(x; z)\Big|_{z\to 0} = \int d^d k\ (z^{-\Delta_+} \phi_+(k; z))\Big|_{z=0} e^{ik\cdot x} C_+(k) = \varphi(x)\,.$$

Therefore, $\Phi_{generic}$ is completely specified by its value at the boundary.

►► **Exercise 8.17:** *Repeat the above argument for the global coordinate case.*

►► **Exercise 8.18:** *How does the above change if we have* Δ_- *mode?*

We are hence led to

$$\Phi_{Bulk}(x; z) = \int d^d y\ G(z; x, y)\ \varphi(y)\,, \tag{8.38}$$

where $G(z;x,y)$ is the boundary-to-bulk propagator:

$$(\Box - m^2)G(z;x,y) = \left(\frac{z}{R}\right)^{d+1} \delta^d(x-y)\delta(z)\,. \tag{8.39}$$

The above has two solutions, specified by $\Delta_\pm$, one of them is not normalizable and the normalizable one satisfies:

$$G(z;x,y)\Big|_{z\to 0} \sim z^{-\Delta_+}\delta^d(x-y)\,.$$

►► **Exercise 8.19:** *One may consider a more general Bulk-to-Bulk propagator* $G(z,z';x,x')$*:*

$$(\Box - m^2)G(z,z';x,x') = \left(\frac{z}{R}\right)^{d+1} \delta^d(x-x')\delta(z-z')\,. \tag{8.40}$$

Since the above equation is $O(d,2)$ *invariant one would expect* $G(z,z';x,x')$ *to be only a function of the* $O(d,2)$ *invariant* ***geodesic*** *distance between to arbitrary points* $(x;z)$ *and* $(x';z')$ *in the* AdS_{d+1}*.*

1. *Show that this geodesic distance* ℓ *is*

$$\cosh\ell = \frac{z^2+z'^2+(x-x')^2}{2zz'}\,. \tag{8.41}$$

2. *The most general form*

$$G = c_+G_+(\ell) + c_-G_-(\ell)\,. \tag{8.42}$$

   Show that

$$G_\pm(\ell) = \frac{2^{-\Delta_\pm}\mathcal{N}_\pm}{2\nu}\cosh\ell^{-\Delta_\pm}\ {}_2F_1\Big(\frac{\Delta_\pm}{2},\frac{\Delta_\pm+1}{2};\pm\nu+1;\cosh^{-2}\ell\Big)\,, \tag{8.43}$$

   where $\Delta_\pm = d/2 \pm \nu$ *with* $\nu = \sqrt{d^2/4+(mR)^2}$, ${}_2F_1$ *is the hypergeometric function and* $\mathcal{N}_\pm$ *is given below* (8.45)*.*

3. *In the* $z'\to 0$*, the above is expected to reduce to boundary-to-bulk propagator discussed above. Show that this indeed happends and*

$$G_{\Delta_\pm}(z;x,y) = \mathcal{N}_\pm\left(\frac{z}{z^2+(x-y)^2}\right)^{\Delta_\pm}\,. \tag{8.44}$$

4. *Show that in* (8.44)

$$\mathcal{N}_\pm = \frac{2\Gamma(\Delta_\pm)}{vol(S^{d-1})\Gamma(\Delta-\frac{d}{2})}\,. \tag{8.45}$$

One may give a more pictorial/Feynman diagram-type description to the above propagators.......

▶▶ **Exercise 8.20:** ***Interactions in the Bulk and the n-point functions at the boundary***. *Let us suppose that we have a scalar bulk field given in* (8.38) *and* (8.44). *Show that*

$$\int d^{d+1}X\Phi_1(X)\Phi_2(X)\Phi_3(X) = C_{123}\frac{1}{|x_1-x_2|^{\Delta_1+\Delta_2-\Delta_3}|x_1-x_3|^{\Delta_1+\Delta_3-\Delta_2}|x_2-x_3|^{\Delta_2+\Delta_3-\Delta_1}} \tag{8.46}$$

where $\Phi_i(X)$ are defined by $\delta(x_i)$ sources at the boundary. What are the coefficients C_{123}? As a reference see hep-th/9804058. The result of the above exercise is a strong test for AdS/CFT where the spacetime dependence of three point function of the CFT is fixed the symmetries. It also makes it clear that three point functions of the "boundary" CFT and the bulk theory are related to each other in an intricate and intriguing way.

8.3.2 On-shell action and holographic regularization

As mentioned in the formal definition of the AdS/CFT or gauge/gravity correspondence (7.4), to compute gauge theory correlators one should compute the on-shell gravity action. Now that we have the explicit form of the AdS solutions we may directly insert it into the action and compute its value. Upon doing so, we find that the integrand (in the expression for the action) is generically divergent in $z = 0$ region and it needs "regularization". The simplest way to get rid of the divergent terms, as we will see, is to drop the divergent parts at $z = \epsilon$ and keep the finite ones.

To gain a better intuition, let us perform computation of on-shell action and the holographic regularization procedure for the scalar field case. We start with the scalar action (8.1)

$$S = \int d^{d+1}x\, \sqrt{-g}\left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - \frac{1}{2}m^2\phi^2 - V(\Phi)\right].$$

As mentioned we will be treating $V(\Phi)$ which contains higher powers of Φ perturbatively. Written on the AdS in Poincaré coordinates the above takes the form

$$S \simeq -\frac{1}{2}\int dzd^dx\, \left(\frac{R}{z}\right)^{d-1}\left[(\partial_z\Phi)^2 + (\partial_\mu\Phi)^2 + m^2\frac{R^2}{z^2}\Phi^2\right]. \tag{8.47}$$

As a warmup for playing with the actions, let us consider the "canonically normalized" field φ,

$$\Phi(x;z) = \left(\frac{R}{z}\right)^{d/2}\varphi(x;z)$$

S takes the form (hereafter for simplicity we will replace z/R by z, or set $R = 1$)

$$S \simeq -\frac{1}{2}\int \frac{dz}{z}d^dx\, \left[(z\partial_z\varphi)^2 + z^2(\partial_\mu\varphi)^2 + \nu^2\varphi^2\right] - \frac{1}{2}\int d^dx\varphi^2\Big|_{z=\epsilon}^{\Lambda}, \tag{8.48}$$

where $\nu^2 = m^2R^2 + d^2/4$ and we have chosen z to range in (ϵ, Λ).

NOTE: *In y coordinate $dy = dz/z$, the φ field is indeed canonically normalized, i.e. the coefficient of $(\partial_y \varphi)^2$ term is one.*

To compute the on-shell action we note that

$$S \simeq \frac{1}{2}\int dz d^d x \left(\frac{R}{z}\right)^{d+1} \left[\Phi(\Box - m^2)\Phi(\partial_\mu \Phi)^2 - \nabla_\mu(\Phi \nabla^\mu \Phi)\right] .$$

The first term vanishes on-shell and assuming that the value of fields at $x^\mu \to \infty$ are falling off fast enough the last term only receives contribution from ∂_z term and hence

$$S_{on-shell} = -\frac{1}{2}\int d^d x \left(\sqrt{-g} g^{zz} \Phi \partial_z \Phi\right)\Big|_{z=\epsilon}^{\gamma} . \tag{8.49}$$

▶▶ **Exercise 8.21:** *Use* (8.38) *and properties of Boundary-to-bulk propagator $G(z; x, y)$ to compute the above on-shell action in terms of sources $\phi(y)$.*

The above on-shell action depends on both on ϵ (UV cutoff) and IR cutoff γ. Since the behavior of the fields at large z close to the Poincaré horizon is not relevant to their UV physics, we may safely choose that on-shell action has no γ dependence and only focus on its ϵ dependence. To this end, recall that any scalar field at small z has the following expansions for two cases with ν parameter being an integer or not:

- Generic non-integer ν

$$\Phi \simeq z^{\Delta_-}\left[(\phi_0(x) + z^2\phi_1(x) + \cdots) + (z^{2\nu}\tilde{\phi}_0(x) + z^{2(\nu+1)}\tilde{\phi}_1(x) + \cdots)\right] . \tag{8.50}$$

- Integer $\nu = n$

$$\begin{aligned}\Phi \simeq z^{\Delta_-}\Big[&(\phi_0(x) + z^2\phi_1(x) + \cdots + z^{2(n-1)}\phi_{n-1}(x) + z^{2n}\ln z\phi_n(x) + \cdots) + \\ &+ (z^{2n}\tilde{\phi}_0(x) + z^{2(n+1)}\tilde{\phi}_1(x) + \cdots)\Big] .\end{aligned} \tag{8.51}$$

It is readily seen that $\phi_n(x)$ functions are all coming from the expansion of K_ν and the $\tilde{\phi}_n$ terms from I_ν (*cf.* (8.29) and (8.30)).

The above expansions are quite generic and one should still impose equations of motion on them. This will relate $\phi_n(x)$ functions to $\phi_0(x)$ and $\tilde{\phi}_n$'s to $\tilde{\phi}_0$.

▶▶ **Exercise 8.22:** *Show that for integer ν case*

$$\phi_k = \frac{1}{2^{2k}}\begin{pmatrix}\nu \\ k\end{pmatrix}(\Box_d)^k \phi_0(x) , \tag{8.52}$$

where $\Box_d$ is d dimensional Laplacian on flat space.

▶▶ **Exercise 8.23:** *Workout a similar relation recursive relation among $\tilde{\phi}_n$'s in terms of $\tilde{\phi}_0$.*

▶▶ **Exercise 8.24:** *Repeat the above analysis when BF bound is saturated.*

Inserting the above into the on-shell action (8.49) we get

$$S_{on-shell} = \frac{1}{2}\int_{b'dry} d^dx\ \epsilon^{-2\nu}(\Delta_-\phi^2 + \epsilon\phi\partial_z\phi)\Big|_{z=\epsilon}\ , \tag{8.53}$$

where

$$\Phi(x;z) = z^{\Delta_-}\phi(x;z)\,. \tag{8.54}$$

As we see, due to presence of negative powers of ϵ, starting from maximum $\epsilon^{-2\nu}$ to $\ln\epsilon$ terms, the above on-shell action is generically divergent and it needs "regularization". We use *minimal subtraction scheme:* We drop terms which come with negative powers of ϵ and keep the first finite terms. Moreover, we may have $\ln\epsilon$ term which will be kept. All these divergent terms will be collected in $S_{c.t.}$ (the counter-terms) and

$$S_{reg.} = S_{on-shell} - S_{c.t.}\,. \tag{8.55}$$

NOTE: *For the cases saturating the BF bound $\nu = 0$ and hence the regularization procedure will be a bit different.*

▶▶ **Exercise 8.25:** *Compute $S_{c.t.}$ and show that it does NOT involve $\tilde{\phi}_n$'s and therefore,*

$$S_{reg} = S_{reg}[\phi_0, \tilde{\phi}_0; \epsilon]\,. \tag{8.56}$$

▶▶ **Exercise 8.26:** *Show that for integer ν case*

$$S_{reg} = \int d^dx\ d\cdot\phi_0(x)[(\tilde{\phi}_0 + \phi_\nu(x)/d) + \phi_\nu(x)\ln\epsilon]\ . \tag{8.57}$$

▶▶ **Exercise 8.27:** *Workout the regularized on-shell action for generic ν. What is the $S_{c.t.}$?*

NOTE: *One may directly compare (8.57) with $\int\lambda\mathcal{O}$ term in the deformed CFT action, identifying λ with ϕ_0.*

Renomalized action

So far we have regularized the on-shell action as a function of the expansion coefficient of the bulk field Φ. But we need to write the action in terms of Φ and other bulk fields to obtain **renormalized action**, explicitly

$$S_{ren.}[\Phi, g_{\mu\nu}] = S_{reg.}\Big|_{\epsilon\to 0}\ . \tag{8.58}$$

In fact, form of $S_{ren.}$ may directly be read from (8.53), noting that $g_{\mu\nu} \sim \epsilon^{-2}$ and that generically Φ has two Δ_+ and Δ_- terms and that $\Delta_+ + \Delta_- = d$.

Having the renormalized action one can now explicitly compute the operator VEV's, using (8.36), e.g.

$$\langle \mathcal{O}(x)\rangle_\mu = -\Delta_- \tilde{\phi}_0(x) + c\phi_\nu(x)\ln\mu \tag{8.59}$$

where the VEV is given at scale μ and c, c' are two numeric coefficients.

▶▶ **Exercise 8.28:** *Work out the coefficients Δ_-, c, c' in the above equation. In particular, note that the coefficient Δ_- (not not just d) has appeared because renormalized action should be written in terms of Φ, this introduces extra numeric factors. For details see [hep-th/0002230, 0209067].*

Holographic renormalization

One may readily find the RG flow equations for the couplings and VEV of operators using the above picture, nothing that

1. RG sliding scale is nothing but a diffeomorphism on holographic direction z (*cf.* discussions in earlier sections): $(z, x^\alpha) \to (\mu z, \mu x^\alpha)$,
2. Φ is scalar in the bulk, $\Phi'(\mu x, \mu z) = \Phi(x, z)$.

Among other things, recalling the low z expansion of Φ implies that

$$\phi_k(\mu x) = \mu^{-(\Delta_-+k)}\phi_k(x)\,, \qquad \tilde{\phi}_k(\mu x) = \mu^{-(\Delta_++k)}\tilde{\phi}_k(x)\,, \tag{8.60}$$

for non-integer ν. When $\nu = n \in \mathbb{Z}$,

$$\begin{aligned} \phi_k(\mu x) &= \mu^{-(\Delta_-+k)}\phi_k(x)\,, \qquad k < n\,, \\ \phi_n(\mu x) &= \mu^{-\Delta_+}\phi_n(x)\,, \\ \tilde{\phi}_0(\mu x) &= \mu^{-\Delta_+}(\tilde{\phi}_0(x) - \ln\mu^2\phi_n(x))\,, \end{aligned} \tag{8.61}$$

▶▶ **Exercise 8.29:** *Find how $\phi_k(\mu x)$ $k > n$ and $\tilde{\phi}_l(\mu x)$ $l > 0$ should scale.*

From the above one can compute RG equations for ϕ_k as well as RG flow for $\langle \mathcal{O}\rangle$:

$$\begin{aligned} \mu\frac{\partial \phi_k(\mu x)}{\partial \mu} &= -(\Delta_- + k)\phi_k(\mu x)\,, \\ \mu\frac{\partial \langle \mathcal{O}(\mu x)\rangle}{\partial \mu} &= \mu^{-\Delta_+}(\langle \mathcal{O}(x)\rangle - 2n\ln\mu^2\phi_n(\mu x))\,. \end{aligned} \tag{8.62}$$

Note that the above has been given for integer $\nu = n$. The last term in the scaling of $\langle \mathcal{O}\rangle$ is the conformal anomaly for this VEV. A part of this term which is proportional to $\mathcal{O}(x)$ term itself may be absorbed into the anomalous dimension of operator $\mathcal{O}$. However, $\phi_n(x)$ is not in general proportional to $\mathcal{O}(x)$ and the piece corresponds conformal anomaly.

▶▶ **Exercise 8.30:** *For the $\phi_n(x) = A\mathcal{O}(x)$ show that the anomalous dimension of operator $\mathcal{O}$ is $2nA$.*

8.3.3 Massless gauge field

As discussed the the details of the solutions of e.o.m on AdS_{d+1} depends on the mass and spin of the field. In particular, let us consider massless gauge fields governed by the action (8.15) when $m=0$. As pointed out for the gauge/gravity correspondence we need to fix $A_z=0$ gauge. It is convenient to introduce the "effective d dimensional gauge field $\mathcal{A}_\mu$ as

$$\mathcal{A}_\mu(x;z)=zA_\mu(x;z)\,,\qquad \mu=0,1,\cdots,d-1\,,$$

where $\mathcal{A}_\mu$ is a d vector whose index is lower and raised by the d dimensional metric g_d (which differs by the constant z surface metric with a factor of z^2). In terms of $\mathcal{A}$ and its field strength $\mathcal{F}_{\mu\nu}$ the action reduces to

$$S=-\frac{1}{4}\int dzd^dx\,\sqrt{-g_d}\;z^{1-d}\left[\mathcal{F}_{\mu\nu}^2+2(\partial_z\mathcal{A}_\mu)^2+2\frac{1-d}{z^2}\mathcal{A}_\mu^2\right]-\int d^dx\,\sqrt{-g_d}(z^{-d}\mathcal{A}_\mu^2)\Big|_{z=\epsilon}\,,\tag{8.63}$$

where the last term is a boundary term. The equation of motion takes the form

$$z^{d-1}\partial_z(z^{1-d}\partial_z\mathcal{A}_\nu)+\frac{d-1}{z^2}\mathcal{A}_\nu+\nabla^\mu\mathcal{F}_{\mu\nu}=0\tag{8.64}$$

where the ∇^μ is a d dimensional derivative and its index is raised by g_d. The consistency condition of the above equation implies the d dimensional transversality condition $\nabla^\mu\mathcal{A}_\mu=0$. Since the above equation is linear in $\mathcal{A}_\mu$, and has d dimensional Poincaré symmetry, one can solve it by

$$\mathcal{A}_\mu(x;z)=z^{d/2}\mathcal{E}_\mu(k;z)\mathrm{e}^{ik\cdot x}\,,\qquad k\cdot\mathcal{E}(k;z)=0\,,\tag{8.65}$$

reducing (8.64) to a Bessel equation for $\mathcal{E}_\mu(k;z)$:

$$\left[\partial_z^2+\frac{1}{z}\partial_z-\left(k^2+\frac{(d/2-1)^2}{z^2}\right)\right]\mathcal{E}_\mu(k;z)=0\,.\tag{8.66}$$

The most general solution to the above for $k^2>0$ is

$$\mathcal{E}_\mu(k;z)=f_\mu^+(k)I_{d/2-1}(kz)+f_\mu^-(k)K_{d/2-1}(kz)\,.\tag{8.67}$$

The above confirms our earlier results that gauge fields have two modes with $\Delta_\pm=d/2\pm(d-2)/2$, i.e. $\Delta_+=d-1,\ \Delta_-=1$.

▶▶ **Exercise 8.31:** *Work out the normalizability and zero-energy through the boundary, conditions and show that Δ_- mode is non-normalizable while Δ_+ is normalizable.*

For performing explicit computations we need the near boundary expansion of the gauge field. The expansion would be different for even d or odd d, due to a difference in behavior of the Bessel functions:

- Even d

$$\mathcal{A}_\mu(x;z)=z\left[(\mathcal{A}_\mu^0(x)+\cdots+z^{d-3}\mathcal{A}_\mu^{d-3}+z^{d-2}\ln z\mathcal{A}_\mu^{d-2}(x)+\cdots)+z^{d-2}(\mathcal{J}_\mu^0(x)+\cdots)\right]\tag{8.68}$$

• Odd d

$$\mathcal{A}_\mu(x;z) = z\left[(\mathcal{A}^0_\mu(x) + \cdots + z^{d-2}\mathcal{A}^{d-2}_\mu(x) + \cdots) + z^{d-2}(\mathcal{J}^0_\mu(x) + \cdots)\right] \tag{8.69}$$

For the odd d case one may use the expansions

$$K_{n+1/2}(z) = \sqrt{\frac{\pi}{2z}}\mathrm{e}^{-z}\ \sum_{k=0}^{n}\frac{(n+k)!}{2^k k!(n-k)!}\ z^{-k}\,,$$
$$I_{n+1/2}(z) = \frac{1}{\sqrt{2\pi z}}\left[\mathrm{e}^{z}\sum_{k=0}^{n}\frac{(-)^k(n+k)!}{2^k k!(n-k)!}\ z^{-k} - (-)^n\mathrm{e}^{-z}\sum_{k=0}^{n}\frac{(n+k)!}{2^k k!(n-k)!}\ z^{-k}\right].$$

One may insert the above mode expansions into (8.63) and compute the on-shell action, carry out the holographic regularization procedure and compute the renormalized on-shell action. Since the analysis is basically the same as those in the scalar case we leave it as an exercise.

▶▶ **Exercise 8.32:** *In the gauge-side theory the coefficients $\mathcal{A}_0$ and $\mathcal{J}_0$ correspond to the deformation of the CFT by the relevant operator $\mathbf{J}^\mu$ which has scaling dimension $d-1$ and the coupling λ_μ, through a term like $\int \mathbf{J}^\mu\lambda_\mu$. The coupling λ_μ may be viewed as an external electric field applied to a system of electrons with current $\mathbf{J}$.*

1. *Compute $\langle \mathbf{J}_\mu\rangle$.*
2. *In general $\langle \mathbf{J}^\mu\rangle$ is a function of the coupling λ_μ. Compute the conductance tensor $\sigma^{\mu\nu} = \frac{\delta \mathbf{J}^\mu}{\delta\lambda_\nu}$.*

8.3.4 Metric perturbations

If we are interested in performing computations with energy-momentum tensor $\mathbf{T}_{\mu\nu}$ of the gauge theory side, e.g. $\langle \mathbf{T}_{\mu\nu}\rangle$ or the trace anomaly $\langle \mathbf{T}_\mu{}^\mu\rangle$ or the central charge $\langle \mathbf{T}_{\mu\nu}\mathbf{T}_{\alpha\beta}\rangle$, we need to study metric perturbation on the AdS$_{d+1}$ side. We start with the Einstein-Hilbert action plus cosm.const. in $d+1$ dimensions and consider metric perturbations as

$$ds^2 = (\bar g_{IJ} + \tilde h_{IJ})dx^I dx^J\,, \qquad I,J = 0,1,\cdots,d, \tag{8.70}$$

where $\bar g_{IJ}$ is the AdS$_{d+1}$ metric in some coordinate system. As discussed in previous sections and as in the vector case, we fix the gauge (diffeomorphisms) such that $h_{Iz}=0$ and choose the convention that $I=(\mu,z)$:

$$ds^2 = \frac{1}{z^2}(dz^2 + h_{\mu\nu}(x;z)dx^\mu dx^\nu)\,, \tag{8.71}$$

One may work out the equation of motion for metric perturbation $h_{\mu\nu}$. We also fix $h_{\mu\nu}$ to be traceless and divergence-free, $h^\mu_\mu = 0$ and $\nabla^\mu h_{\mu\nu}=0$.

▶▶ **Exercise 8.33:** *Show that the above conditions are equivalent to $\nabla^I h_{IJ}=0$, $h_I{}^I = 0$.*

▶▶ **Exercise 8.34:** *Work out the equations of motion for the perturbations $h_{\mu\nu}$.*

One may solve these equations through

$$h_{\mu\nu}(x;z) = z^{d/2}\mathfrak{h}_{\mu\nu}(k;z)\mathrm{e}^{ik\cdot x}\,, \qquad k^\mu\mathfrak{h}_{\mu\nu}(k;z)=0\,,\ \mathfrak{h}^\mu{}_\mu=0\,, \tag{8.72}$$

where $_{\mu\nu}(k;z)$ satisfies a Bessel equation:

$$\left[\partial_z^2+\frac{1}{z}\partial_z-\left(k^2+\frac{(d/2)^2}{z^2}\right)\right]\mathfrak{h}_{\mu\nu}(k;z)=0\,. \tag{8.73}$$

The most general solution to the above for $k^2>0$ is

$$\mathfrak{h}_{\mu\nu}(k;z)=f^+_{\mu\nu}(k)I_{d/2}(kz)+f^-_{\mu\nu}(k)K_{d/2}(kz)\,. \tag{8.74}$$

The above confirms our earlier results that massless spin two (gravitons) have two modes with $\Delta_\pm=d/2\pm d/2$, i.e. $\Delta_+=d,\ \Delta_-=0$. That is, deformation of the CFT by $\mathbf{T}_{\mu\nu}$ is a marginal deformation with coupling $h^{\mu\nu}$.

▶▶ **Exercise 8.35:** *Which of these modes is noramlizable and which non-normalizable?*

As in the gauge field case the near boundary expansion of the metric would be different for even d or odd d:

- Even d

$$h_{\mu\nu}(x;z)=(h^{(0)}_{\mu\nu}(x)+\cdots+z^{d-1}h^{(d-1)}_{\mu\nu}+z^d\ln z\ h^{(d)}_{\mu\nu}(x)+\cdots)+z^d(\mathfrak{T}^{(0)}_{\mu\nu}(x)+\cdots)\,, \tag{8.75}$$

- Odd d

$$h_{\mu\nu}(x;z)=(h^{(0)}_{\mu\nu}(x)+\cdots+z^dh^{(d)}_{\mu\nu}(x)+\cdots)+z^d(\mathfrak{T}^{(0)}_{\mu\nu}(x)+\cdots) \tag{8.76}$$

For the odd d case one may use the expansions given in (8.70).

The above, after insertion into (8.71), constitute the **near boundary Fefferman-Graham expansion for metric**.

▶▶ **Exercise 8.36:**

1. *Compute the on-shell Einstein-Hilbert action expanded up to second order in $h_{\mu\nu}$.*
2. *This action is expected to only involve logarithmic divergences for even d which correspond to trace anomaly, compute this.*
3. *Compute the two-point function of two energy momentum tensors in the gauge theory side using this renormalized on-shell second order action.*

8.4 Wilson loops in AdS/CFT

As pointed out local operators of the gauge theory side and their correlators may be computed using (renormalized) on-shell gravity action with the boundary field values associated with the couplings of the operators in the gauge theory side. The gauge theory, however, has other *non-local* gauge invariant observables, the **Wilson or Polyakov loops**:

$$\mathcal{W}_C = \text{Tr}\left(\text{P e}^{i\oint_C A\cdot dx}\right) . \tag{8.77}$$

For supersymmetric gauge theories one can introduce a "supersymmetrized Wilson loop," where in the exponent besides the gauge field we also have superpartners of the gauge field, including gauginos and possibly scalar fields in the gauge multiplet. The VEV of super-Wilson loops can preserve some part of the supersymmetry, depending on C; e.g. circular Wilson loops in $\mathcal{N} = 4$ SYM preserve half of the SUSY, less symmetric C will preserve less SUSY. There has been an elaborate study of such Wilson loops and their classification by the amount of SUSY they preserve.

Wilson loop, once the collection of all possible loops/paths is considered provide an (over)complete basis for the Hilbert space, which is usually spanned by local "gluon-type" states. Moreover, VEV of Wilson loops in a theory with charges/quarks and for loops which consist of two long time-like (light-like) legs and two (comparatively) short space-like legs, may be used to compute quark-antiquark potential, even in non-perturbative regime. Explicitly, let us consider a loop C of in (t, x) plane consisting of two parallel line along the time direction separated in the x direction by distance L, then

$$V(L)T = iq \ln(\langle \mathcal{W}_C(L)\rangle) . \tag{8.78}$$

This loop is essentially the path of two quarks or anti-quarks separated by distance L, $V(L)$ is the potential between them and q is their charge and T is the length of their worldline.

One would wonder if gauge/gravity correspondence or AdS/CFT duality provides a way for computing Wilson loops. To this end, we need to go beyond gravity limit and consider string theory. As discussed D-branes are a place where open string end points can attach. Moreover, we argued that in the near horizon limit D-branes are replaced by flux and leave a space with boundary. It is hence natural to assume that open strings can end on the boundary in the same way they attached D-branes. One can show that this indeed provides a consistent setup. One way to see this is to consider a D3-brane system consisting of $N+1$ branes, a stack of N coincident branes and one separated from them. Taking the near-horizon limit while also tuning the distance between the two stacks to zero appropriately, we end up with and AdS_5 space (in Poincaré coordinates) and a D3-brane in AdS space. There were open strings stretched between the two stacks (W-bosons of the Higgsing which break $U(N+1)$ to $U(N)\times U(1)$.) These open strings are in the fundamental and anti-fundamental representations of the $U(N)$, and hence behave like quarks and anti-quarks of the dual gauge theory. Therefore, we have the setup very similar to the one depicted in the previous paragraph and we may hence use these open strings for computing Wilson loops.

Proposal: $\ln(\langle \mathcal{W}_C\rangle)$ is equal to the extremum of the string worldsheet action for open string worldsheet Σ such that projection of Σ on the boundary is bounded by the closed path C. Explicitly:

$$\ln(\langle \mathcal{W}_C \rangle) = S_{string}[\Sigma], \quad \textit{such that} \quad C = \partial(\Sigma\ \cap \partial \text{AdS})\,, \tag{8.79}$$

and Σ should minimize S_{string} or satisfy worldsheet equations of motion, and

$$S_{string} = \frac{1}{4\pi\alpha'}\int_\Sigma d\tau d\sigma\ \sqrt{\det\left(G_{IJ}\partial_a X^I \partial_b X^J\right)}\,, \qquad I, J = 0, 1, \cdots, 9;\ a, b = (\tau, \sigma)\,, \tag{8.80}$$

and G_{IJ} is the (asymptotically) AdS metric.

▶▶ **Exercise 8.37:** *Extract out this proposal directly from the expression for the AdS/CFT duality* (7.3)*, in appropriate "saddle point approximation."*

Let us now perform the first explicit computation with the above proposal: consider C which corresponds to static quark and anti-quarks with separation L. We choose the remaining part of worldsheet diffeomorphism invariance to fix

$$x^0 = \tau\,, \qquad x^1 = \frac{L}{\pi}\sigma\,, \qquad z = z(\sigma)\,, \qquad \text{other X's} = 0\,. \tag{8.81}$$

We choose the boundary condition: $z \to 0$, to correspond to $\sigma = 0, \pi$ such that $x^1 = 0, L$ at the boundary. Note also that we have considered a "static" string configuration. For this configuration,

$$\begin{aligned} S &= \frac{R^2}{4\pi\alpha'} \cdot \frac{\pi}{L}\int d\tau \int_0^\pi d\sigma\ \frac{1}{y^2}\sqrt{1 + y'^2} \\ &= T\frac{\sqrt{\lambda}}{4L}\int_0^\pi d\sigma\ \frac{1}{y^2}\sqrt{1 + y'^2} \end{aligned} \tag{8.82}$$

where $z = \frac{L}{\pi}y$, $T = \int d\tau$, $\lambda = R^4/\alpha'^2$ is the t' Hooft coupling and $y' = \frac{dy}{d\sigma}$.

The second line in (8.82) is an action for $y(\sigma)$ and the Lagrangian $\mathcal{L}$ is independent of σ variable, therefore,

$$y'\frac{\partial \mathcal{L}}{\partial y'} - \mathcal{L} = const. \qquad \Longrightarrow \qquad y^2\sqrt{1 + y'^2} = y_0^2. \tag{8.83}$$

- From the above we can already see that in near boundary region $y \to 0$, $y' \sim y^{-2} \to \infty$; i.e. the open string is attached *orthogonally* to the boundary. This is consistent with our open strings attached to D-branes with Dirichlet boundary conditions discussed above.
- The value of constant is chosen such that y_0 is the value of y when $y' = 0$; y_0 is the maximum value y reaches to before turning again to $y = 0$ at $\sigma = \pi/2$.

From (8.83) we learn that

$$\int_0^{y_0} \frac{y^2 dy}{\sqrt{y_0^4 - y^4}} = y_0 \int_0^1 \frac{x^2 dx}{\sqrt{1 - x^4}} = \frac{\pi}{2}\,, \tag{8.84a}$$

$$S = 2T\frac{\sqrt{\lambda}}{4L}\int_0^{y_0} \frac{dy}{y^2\sqrt{y_0^4 - y^4}} = 2T\frac{\sqrt{\lambda}}{4L}\frac{1}{y_0^3}\int_0^1 \frac{dx}{x^2\sqrt{1 - x^4}}\,. \tag{8.84b}$$

As we see y_0 is completely fixed by the (8.84a), it is just a fixed number. The integral in (8.84a) may be computed in terms of a standard complete elliptic integral.

▶▶ **Exercise 8.38:** *Show that* $y_0 = \frac{\Gamma(\frac{1}{4})^2}{2\sqrt{2\pi}}$.

From (8.84b), however, we can see two points:

- The action S is infinite, since the integral has a divergent piece around $y = 0$ region;
- all parametric dependence of action is in the coefficient in front and in particular its dependence on the t' Hooft coupling is of the form $\sqrt{\lambda}$ and its dependence on the separation of quarks L is of the form $1/L$.

Since the action is divergent we need to regularize it. As argued, our picture is that as if the quark anti-quark system is coming from the near-horizon limit over two stacks of branes where one of the is now sent to the boundary ($y = 0$) and the other to $y = \infty$. Therefore, we are dealing with a very massive quark system. If we are interested in the *interaction energy* of the quarks, we need to subtract off the quark mass contribution, explicitly:

$$S_{reg.} = 2T\frac{\sqrt{\lambda}}{4L}\frac{1}{y_0^3}\left[\int_0^1 \frac{dx}{x^2\sqrt{1-x^4}} - \int_0^\infty \frac{dx}{x^2}\right] \tag{8.85}$$

The value of the integral, whatever it is is a positive number let us call it $2c$. Therefore,

$$V_{Q-\bar{Q}} = -c\frac{\sqrt{\lambda}}{L}. \tag{8.86}$$

▶▶ **Exercise 8.39:** *Show that the coefficient* $c = \frac{4\pi^2}{(\Gamma(\frac{1}{4}))^4}$.

The above result has two features which are notable:

- Its L dependence is $1/L$, in accord with conformal invariance of the SYM theory, recalling the fact that there is not other dimensionful parameter in the game. In particular, note that this result is independent of the dimension of AdS space (or the dual CFT).
- Its coupling dependence is only through t' Hooft coupling, as expected. Nonetheless, this dependence is not analytic in the coupling and involves $\sqrt{\lambda}$, which is related to the point that our gauge theory is strongly coupled.

▶▶ **Exercise 8.40:** *Compare this with result with the usual Coulomb potential between an electron and a positron.*

▶▶ **Exercise 8.41:** *Repeat the above analysis and compute the potential between two quark and anti-quarks separated in the x-direction while moving relative to each other along the y-direction with velocity v.*

▶▶ **Exercise 8.42:** *Find the quark anti-quark potential in a non-zero temperature gauge theory, corresponding to AdS-blackbrane background with metric*

$$ds^2 = \frac{R^2}{z^2}\left[-f(z)dt^2 + d\vec{x}^2 + \frac{dz^2}{f(z)}\right], \qquad f(z) = 1 - \frac{z^4}{z_0^4}. \tag{8.87}$$

.

▶▶ **Exercise 8.43:** *Find the quark-antiquark potential in the confining background with metric:*

$$ds^2 = \frac{R^2}{z^2}\left[-dt^2 + dx_1^2 + dx_2^2 + f(z)dy^2 + \frac{dz^2}{f(z)}\right], \tag{8.88}$$

with $f(z)$ given in (8.87)*. Suppose that quarks are separated in x_1 direction.*
Note that the confining background may be obtained from the black brane solution (8.87) *upon Wick rotating along t and Wick rotating back along x_3.*

9 Black holes in AdS and non-zero temperature QFT

One of the questions which arises in QFT's is studying physical processes in a non-zero temperature. Here we study how gauge/gravity correspondence relates the two. We first introduce the non-zero temperature field theory and then argue that in the gravity side temperature corresponds to introducing black hole in the bulk of AdS. We are hence led to statement of gauge/gravity duality at non-vanishing temperature. Finally, we show some basic checks of this proposal through computing the partition function on both sides.

9.1 Non-zero temperature QFT

As usual all the information may be extracted from the partition function. In any stat.mech. system at temperature $T = 1/\beta$

$$\mathcal{Z} = \mathrm{Tr}e^{-\beta\mathbf{H}}, \tag{9.1}$$

where $\mathbf{H}$ is the Hamiltonian of the system and trace is over all states/configurations in the Fock space and may hence also be represented by the path integral. One may define a normalized thermal density matrix $\hat{\rho}_{\beta}$

$$\hat{\rho}_{\beta} = \frac{1}{\mathcal{Z}}\int [\mathcal{D}\Phi]_t \; |\Phi(t,x)\rangle\langle\Phi(t,x)|e^{-\beta\mathbf{H}}. \tag{9.2}$$

Note that the Hamiltonian $\mathbf{H}$ is basically the same Hamiltonian which is generator of time translations. Therefore,

$$
\begin{aligned}
\langle \mathcal{O}_1(t,x_1)\mathcal{O}_2(t,x_2)\cdots\mathcal{O}_n(t,x_n)\rangle_\beta &= \text{Tr}\ \Big(\hat{\rho}_\beta \mathcal{O}_1(t,x_1)\mathcal{O}_2(t,x_2)\cdots\mathcal{O}_n(t,x_n)\Big) \\
&= \int [\mathcal{D}\Phi]_t\ \langle\Phi(t,x)|\mathcal{O}_1(t,x_1)\mathcal{O}_2(t,x_2)\cdots\mathcal{O}_n(t,x_n)\mathrm{e}^{-\beta\mathbf{H}}|\Phi(t,x)\rangle \\
&= \int [\mathcal{D}\Phi]_t\ \langle\Phi(t,x)|\mathcal{O}_1(t,x_1)\mathcal{O}_2(t,x_2)\cdots\mathcal{O}_n(t,x_n)|\Phi(t+i\beta,x)\rangle\,,
\end{aligned}
\tag{9.3}
$$

The above shows the way to make non-zero temperature calculations:

- Wick rotate and replace $t = it_E$;
- Make the Euclidean time periodic $t_E \equiv t_E + \beta$; *i.e.* compactification of Euclidean time amounts to having a non-zero temperature.
- Impose appropriate periodicity conditions; *i.e.* periodicity (in Euclidean time) for bosonic fields/operators and anti-periodicity for fermionic ones.

NOTE: $[\mathcal{D}\Phi]_t$ *in* (9.3) *denote all field configurations* ***at a given time slice*** t*. This path integral does not compute time evolution. System is supposed to be in thermal equilibrium and there is no time evolution. Moreover, all operators are multiplied at the same given time* t*. Finally, it is supposed that Hamiltonian of the system is conserved and has hence no "time" dependence.*

Thermo-Field Double. Instead of working with a *mixed state* density matrix (9.2) to perform QFT analysis one may use Thermo-Field Dynamics (TFD) where the mixed state is replaced by a pure state through doubling the Hilbert space.

`For a review on the TFD e.g. see N. P. Landsman and C. G. van Weert, Real and Imaginary Time Field Theory at Finite Temperature and Density, Phys. Rept. 145 (1987) 141`
`Y. Takahashi and H. Umezawa, Thermo field dynamics, Int.J.Mod.Phys. B10 (1996) 1755.`

Here we just illustrate the doubling and "purification" trick, which has a quite natural appearance in the AdS/CFT setups, in eternal AdS black holes. Let us denote by $\mathcal{H}$ the set of all states $|\Phi(t,x)\rangle$. The Thermo-field doubled Hilbert space is then defined as two copies of $\mathcal{H}$:

$$\mathcal{H}_{TFD} = \mathcal{H}_I \otimes \mathcal{H}_{II},$$

The thermo-field Hamiltonian is then defined as

$$\mathbf{H}_{TF} = \mathbf{H}\otimes \mathbb{1}_{II} - \mathbb{1}_I \otimes \mathbf{H}\,, \tag{9.4}$$

The mixed density matrix (9.2) may now be represented through a pure state $|\Omega\rangle$:

$$|\Omega\rangle = \frac{1}{\sqrt{\mathcal{Z}}}\ \int [D\Phi]_t \mathrm{e}^{-\frac{\beta}{2}\mathbf{H}\otimes\mathbb{1}}|\Phi(t,x)\rangle\otimes|\Phi(t,x)\rangle \tag{9.5}$$

Note that the thermo-field Hamiltonian $\mathbf{H}_{TF}$ has been constructed such that $\mathbf{H}_{TF}|\Omega\rangle = 0$; i.e. $|\Omega\rangle$ is the vacuum state of the thermo-field dynamics theory.

►► **Exercise 9.1:** *Show that $\mathbf{H}_{TF}$ defines a unitary dynamics on $\mathcal{H}_{TFD}$.*

►► **Exercise 9.2:** *Show that the mixed density matrix $\hat{\rho}$ can be obtained through "purified density matrix"*

$$\hat{\rho}_{_\beta} = Tr_{II}\ \hat{\rho}_{\Omega}\,, \qquad \hat{\rho}_{\Omega} = |\Omega\rangle\langle\Omega|, \tag{9.6}$$

where trace is over the second copy of the Hilbert space.

In other words, states in $\mathcal{H}_{II}$ which appear in $|\Omega\rangle$ are maximally thermally entangled with those of $\mathcal{H}_I$. However, the doubled Hilbert space in principle has a bigger set of observables; its observables consists of any observable in I and any observable in II.

One may also compute thermal n-point function in (9.3)

$$\langle \mathcal{O}_1(t,x_1)\mathcal{O}_2(t,x_2)\cdots\mathcal{O}_n(t,x_n)\rangle_\beta = \mathrm{Tr}_{\ \mathcal{H}_I}\left(\hat{\rho}_{_\beta}\mathcal{O}_1(t,x_1)\mathcal{O}_2(t,x_2)\cdots\mathcal{O}_n(t,x_n)\right)\,, \tag{9.7}$$

where $\mathcal{O}_i$ are operators in $\mathcal{H}_I$. However, the same correlation function may be computed with a pure density matrix on the doubled Hilbert space. Explicitly:

$$\begin{aligned} G_n(x_1,x_2,\cdots,x_n;\beta) &= \langle \mathcal{O}_1(t,x_1)\mathcal{O}_2(t,x_2)\cdots\mathcal{O}_n(t,x_n)\rangle_\beta \\ &= \mathrm{Tr}_{\ \mathcal{H}_{TFD}}\left(\hat{\rho}_\Omega\mathcal{O}_1(t,x_1)\mathcal{O}_2(t,x_2)\cdots\mathcal{O}_n(t,x_n)\right) \\ &= \langle\Omega|\mathcal{O}_1(t,x_1)\mathcal{O}_2(t,x_2)\cdots\mathcal{O}_n(t,x_n)|\Omega\rangle \end{aligned} \tag{9.8}$$

$$\begin{aligned} G_n(x_1,x_2,\cdots,x_n;\beta) &= \int[\mathcal{D}\Phi]_t\ [\mathcal{D}\Psi]_t\ \langle\Phi(t_E,x)|\mathcal{O}_1(t,x_1)\cdots\mathcal{O}_l(t,x_l)|\Psi(t_E+\beta,y)\rangle\langle\Phi(x)|\Psi(y)\rangle \\ &= \int[\mathcal{D}\Phi]_t\ \langle\Phi(t_E,x)|\mathcal{O}_1(t,x_1)\cdots\mathcal{O}_l(t,x_l)|\Phi(t_E+\beta,x)\rangle \end{aligned} \tag{9.9}$$

That is, a "vacuum amplitude" in the thermo-field theory (9.8) is equal to a thermal field theory correlator (9.9). This is why the TFD provides a simple and nice description and tool for non-zero temperature field theory computations.

►► **Exercise 9.3:** *Show that*

$$\langle\Omega|\cdots[\mathcal{O}_I(x,t),\mathcal{O}_{II}(y,t)]\cdots|\Omega\rangle = 0,$$

where $\mathcal{O}_I$ and $\mathcal{O}_{II}$ are respectively defined in $\mathcal{H}_I$ and $\mathcal{H}_{II}$.

9.2 Black holes on AdS

Since AdS has a causal boundary, light-rays can reach the boundary and bounce back in a finite time. Therefore, unlike the flat space, black holes on AdS can reach to a thermal equilibrium with their own Hawking radiation, we seem to able to produce eternal AdS black holes. As a result, the "dual" QFT which resides on the boundary of an eternal AdS black hole should be a non-zero temperature QFT at exactly the same temperature as the Hawking temperature of black hole.

This picture dovetails perfectly with the trick for reading the Hawking temperature of a generic black hole: *expand the metric close to the horizon and require that the Euclidean near-horizon metric does not have a conical deficit. The periodicity of Euclidean time is inverse of temperature* β. Let us examine this for two cases:

AdS-Schwarzchild.

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{d-1}^2\,, \qquad f = 1 + \frac{r^2}{R^2} - \frac{2GM}{r^{d-3}} \tag{9.10}$$

If $f(r_h) = 0$, then $f(r) \simeq f'(r_h)(r - r_h) + \mathcal{O}((r - r_h)^2)$ and hence the near horizon metric is of the form

$$ds^2 = -\left(\frac{f'(r_h)}{2}\right)^2 \rho^2 dt^2 + d\rho^2 + r_h^2 d\Omega_{d-1}^2\,, \qquad \rho^2 = \frac{4}{f'(r_h)}(r - r_h)\,. \tag{9.11}$$

Absence of conical singularity in the Euclidean time sector implies that

$$t_E \equiv t_E + \frac{4\pi}{f'(r_h)} \qquad \Longrightarrow \qquad \beta = \frac{4\pi}{f'(r_h)}\,. \tag{9.12}$$

NOTE: *One could have read the temperature directly from* (9.11) *without Wick rotations and going to the Euclidean time, noting that the tr part of the above metric is a Rindler space with a specified acceleration; the Unruh temperature for this space should be the same as the Hawing temperature for black hole. This latter is necessitated by Equivalence Principle.*

►► **Exercise 9.4:** *Compute the Hawking temperature explicitly for the AdS-Sch'ld black hole.*

For $d = 4$ case AdS-Sch'ld black hole is the background dual to a thermal $\mathcal{N} = 4$ SYM on $\mathbb{R}\times\mathrm{S}^3$ at temperature $1/\beta$. In Euclidean signature, this is the SYM on $\mathrm{S}^1\times\mathrm{S}^3$, where the temperature is in fact the ratio of S^1 to S^3 radii.

AdS-black brane. This is the background dual to thermal QFTs on $\mathbb{R}^d$. Let us start with the black brane on AdS_{d+1}(6.10)

$$ds^2 = \frac{R^2}{z^2}\left[f(z)dt_E^2 + d\vec{x}^2 + \frac{dz^2}{f(z)}\right]\,, \qquad f(z) = 1 - \frac{z^d}{z_h^d}\,, \tag{9.13}$$

where the horizon is sitting at $z = z_h$.

►► **Exercise 9.5:** *Compute the temperature using absence of conical singularity in near-horizon region, and show that*

$$T_H = \frac{d}{4\pi z_h}\,. \tag{9.14}$$

9.3 Gauge/gravity at non-zero temperature

Given the above introductory reviews, we are now ready to give a more explicit statement of gauge/gravity correspondence at non-zero temperature:

$$\boxed{Z_{CFT}[\lambda_i(x,\mu_0);\beta] \simeq e^{iS^{Thermal-AdS-backgrounds}_{on-shell}}\Big|_{\lambda_i(x,y_0)}.} \tag{9.15}$$

This statement is obviously a refinement of the more general statement given is section 7.1, (7.4) for this case.

We are here prescribed to compute the on-shell gravity action for fields living on an appropriately chosen "Thermal-AdS-background." This background is generically (not always, see below) AdS black hole/brane background, *at Hawking temperature equal to the desired CFT temperature.*

Using (9.15) one may compute thermal field theory correlators by differentiating the LHS w.r.t. couplings λ_i, which corresponds to making similar derivative on the RHS.

►► **Exercise 9.6:** ***Gauge theory dual of the Hawking-Page transition.*** *As discussed having black holes/branes and existence of event horizon leads to a natural temperature: Near horizon metric in the Euclidean sector involves a part which looks like a 2d flat space written in polar coordinates with Euclidean time being its angular variable, from where we read the temperature. One could have instead compactified Euclidean time direction of the Euclidean AdS on a circle to get a* ***thermal AdS****.*
As discussed in detail in section 7, gauge/gravity correspondence is a saddle point approximation to the "exact" AdS/CFT duality. One may then wonder for a given temperature T which of thermal AdS or AdS black hole dominate the saddle point. To this end, one should compute the Euclidean action on either of these backgrounds and the smaller one for a given temperature is the one dominating the saddle point.
Perform this computation and find the critical temperature T_c where the value of Euclidean action for the two becomes equal.
As always in QFTs, change in the saddle point for making perturbations corresponds to a phase transition. The thermal-AdS to AdS black hole transition in the gravity side is called Hawking-Page transition. According to gauge/gravity, this should hence correspond to a phase transition in gauge theory. Argue that this is a ***confinement-deconfinement*** *phase transition.* `[Ref. E. Witten '1998]`.

9.3.1 Some basic checks

As a basic check for the proposed correspondence (9.15) we compute partition function on both sides, i.e. when all λ_i are turned off. On the gauge theory side this corresponds to performing the Tr in (9.1) and in the gravity side, that computing on-shell AdS gravity action for the AdS-black brane solution.

Gauge theory partition function. As discussed the gauge gravity correspondence relates a strongly coupled gauge theory to a classical AdS gravity. In general we do not know how to compute the partition function for a strongly coupled gauge theory, with the Hamiltonian $\mathbf{H}$ in (9.1) being its Euclidean action. However, we can compute this partition function for a weakly coupled field theory and hope/think that this remains valid at strong coupling regime. This hope is more reasonable for a CFT like $\mathcal{N}=4$ SYM which because of the large amount of supersymmetry many quantities (which unfortunately partition function is not among them) are protected.

The partition function for a d dimensional relativistic (massless) bosonic or fermionic mode is given as

$$\ln \mathcal{Z} = \pm V_{d-1} \int \frac{d^{d-1}p}{(2\pi)^{d-1}} \ \ln\left(1 \pm \mathrm{e}^{-\beta E(p)}\right) \,, \qquad E(p) = |p| \tag{9.16}$$

where plus sign corresponds to fermions and minus to bosons. If we make the change of variables $p = Tx$, in terms of dimensionless variable x we have

$$\ln \mathcal{Z} = C_{\pm} V_{d-1} \cdot T^{d-1} \,, \tag{9.17}$$

where

$$C_{\pm} = \pm \frac{\mathrm{vol}(S^{d-2})}{(2\pi)^{d-1}} \int_0^\infty dx\ x^{d-2} \cdot \ln\left(1 \pm \mathrm{e}^{-x}\right) \,. \tag{9.18}$$

The above integrals, upon an integration by-part and for even d, may be written in terms of Bernouilli numbers B_n , e.g.

$$\int_0^\infty dx\ \frac{x^{2n-1}}{\mathrm{e}^x + 1} = \frac{2^{2n-1}-1}{2n} \pi^{2n} B_n \,, \qquad \int_0^\infty dx\ \frac{x^{2n-1}}{\mathrm{e}^x - 1} = \frac{\pi^{2n}}{4n} B_n \,. \tag{9.19}$$

For a free theory with N_b bosonic and N_f fermionic degrees of freedom, when then have

$$\ln \mathcal{Z}_{Total} = (C_+ N_f + C_- N_b) V_{d-1} \cdot T^{d-1} \,. \tag{9.20}$$

As we see,

$$\ln \mathcal{Z}_{Total} \propto V_{d-1}\ T^{d-1} \,. \tag{9.21}$$

From the above one can deduce,

- $\ln \mathcal{Z}$ is proportional to the volume of space the field theory is defined on, V_{d-1};
- using standard thermodynamical relations, one can compute entropy density $\mathcal{S}$, pressure P and the energy density $\mathcal{E}$:

$$\begin{aligned}
\mathcal{S} &= \frac{\partial}{\partial T}\left(T \ln \mathcal{Z}_{Total}\right) = (C_+ N_f + C_- N_b) \cdot d \cdot T^{d-1} \,, \\
\mathcal{S} &= \frac{\partial P}{\partial T} \qquad \Longrightarrow \qquad P = (C_+ N_f + C_- N_b) \cdot T^d \,, \\
\mathcal{E} &= -P + T\mathcal{S} = (d-1)P \,.
\end{aligned} \tag{9.22}$$

These are the behavior expected from any d dimensional conformal field theory.

▶▶ **Exercise 9.7:** *Show that for free $\mathcal{N}=4$ 4d $U(N)$ SYM,*

$$C_- = \frac{\pi^2}{90}\,, \qquad C_+ = \frac{7\pi^2}{720}\,, \tag{9.23}$$

and $N_b = N_f = 8N^2$, and hence $\mathcal{S} = \frac{2\pi^2}{3}N^2T^3$.

On shell gravity action. To compute the partition function of the QFT side, (9.15) prescribes us to evaluate on-shell gravity action on the AdS-Sch'd (or thermal AdS):

$$S_{gravity} = -\frac{1}{16\pi G_N}\int d^dxdz\sqrt{-g}(R-2\Lambda) = \frac{d(d-1)}{\ell_P^{d-1}R^2}\int dz\left(\frac{R}{z}\right)^{d+1}\int d^dx\,, \tag{9.24}$$

where we have used $8\pi G_N = \ell_p^{d-1}$. The z integral is divergent in near boundary region (at $z=0$) and should be regularized. This regularization is very similar to the one carried out in section 7.2 and we do not repeat it here. This leads to "standard" black hole thermodynamical relations:

$$\begin{aligned}\mathcal{S}_{BH} &= \frac{A_H}{4G_N}\cdot\frac{1}{V_{d-1}}, \qquad A_H = \left(\frac{R}{z_h}\right)^{d-1}\cdot V_{d-1}\,,\\ &= 2\pi\left(\frac{4\pi R}{d\cdot\ell_P}\right)^{d-1}T^{d-1}\,, \qquad 8\pi G_N \equiv \ell_P^{d-1}\,.\end{aligned} \tag{9.25}$$

▶▶ **Exercise 9.8:** *Complete the steps of regularization.*

For a generic thermodynamical system

$$\frac{1}{V_{d-1}}dG = dP - \mathcal{S}dT\,, \tag{9.26}$$

where G is the Gibbs free energy and P is the pressure. For "free" black brane system, Gibbs free energy is a constant and hence

$$\mathcal{S}_{BH} = \frac{\partial P_{BH}}{\partial T} \quad\Longrightarrow\quad P_{BH} = \frac{1}{2}\left(\frac{4\pi}{d}\right)^d\left(\frac{R}{\ell_P}\right)^{d-1}T^d\,. \tag{9.27}$$

Likewise, one can compute the energy density $\mathcal{E}_{BH}$, recalling that $G/V = \mathcal{E} - T\mathcal{S} + P$ and hence

$$\mathcal{E}_{BH} = -P_{BH} + T\mathcal{S}_{BH} = (d-1)P_{BH}\,, \tag{9.28}$$

which is the expected result for a d dimensional system with an energy momentum tensor with vanishing trace; i.e. a conformal invariant system.

The qualitative features of these relations are exactly the same as those in QFT, (9.22). For the exact matching, however, we need to specify the relation between R/ℓ_P and number of degrees of freedom of the dual field theory. Recalling discussions in section 7.2 and keeping track of all numeric factors, for $d=4$ case, we end up with

$$\mathcal{S}_{free-\mathcal{N}=4,SYM} = \frac{4}{3}\mathcal{S}_{BH}\,. \tag{9.29}$$

The mismatch in the numerical factor may be attributed to the fact that the gravity side is indeed computing the entropy for a strongly coupled CFT while the RHS is the entropy of the weakly coupled CFT. The two need not really match up to the numeric details.